PROBLEMS

Individual part

Problem A (Algebra & Combinatorics).
Let $(R, +, \cdot)$ be a commutative ring. If I and J are two ideals of R then prove that

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

and find n such that

$$n\,\mathbb{Z} \;=\; \sqrt{8\,\mathbb{Z}} \cap \sqrt{11\,\mathbb{Z}} \cap \sqrt{2024\,\mathbb{Z}}.$$

Remark
The radical $\sqrt{I}$ of an ideal I is an ideal which consists of all elements in the ring with some power in I, i.e.

$$\sqrt{I} = \{a \in R : \underset{n \geqslant 1}{\exists}\; a^n \in I\}\,.$$

Problem C (Calculus & Mathematical Analysis).
Prove that

$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} \notin \mathbb{Q}.$$

Problem E (Equations & Inequalities).
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(A \triangle B) = f(A) \triangle f(B),$$

where $\triangle$ is the symmetric difference of sets: $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Remark
We define $f(A) = \{x \in \mathbb{R}\colon \underset{a \in A}{\exists}\; f(a) = x\}$ for any subset $A \subset \mathbb{R}$.

Problem G (Geometry & Linear Algebra).
Let x_n denote the maximal determinant of an $n \times n$ matrix with entries equal to ± 1. Does the sequence $\sqrt[n]{x_n}$ have a finite limit?

Problem P (Set Theory & Probability).
Let $(\mathrm{X}, \preccurlyeq)$ be a partially ordered set such that for all subsets $A, B \subset \mathrm{X}$ the following property is satisfied

$$\left(\underset{\substack{x \in A \\ y \in B}}{\forall}\; x \preccurlyeq y \right) \Longrightarrow \underset{z \in \mathrm{X}}{\exists} \left(\left(\underset{x \in A}{\forall}\; x \preccurlyeq z \right) \wedge \left(\underset{y \in B}{\forall}\; z \preccurlyeq y \right) \right).$$

(i) Show that every order preserving function $f\colon \mathrm{X} \to \mathrm{X}$ (i.e. $\underset{x,y \in \mathrm{X}}{\forall}\; x \preccurlyeq y \Rightarrow f(x) \preccurlyeq f(y)$) has a fixed point (i.e. there is an $x_0 \in \mathrm{X}$ such that $f(x_0) = x_0$).

(ii) Give an example of X and f, where the property is satisfied only for non-empty subsets A, B of X and f has no fixed point.

Problem A.1.
Find all $n \in \mathbb{N}^+$ such that there exist two cycles c_1, c_2 of length n in S_n, such that c_1 and c_2 are the generators of S_n.
Remark
Generators for a group G is a subset of elements $S \subset G$ such that every element of G can be expressed as a combination (using the group operation) of finitely many elements from S and their inverses. The smallest subgroup of G that contains $\{g_1, \ldots, g_n\}$ is denoted $\langle g_1, \ldots, g_n \rangle$, and if $\langle g_1, \ldots, g_n \rangle = G$, we say that $\{g_1, \ldots, g_n\}$ generates G.

Problem A.2.
Let p and q be distinct prime numbers. Let G be a group of order $p^3 \cdot q$ such that its commutator subgroup K is of order q. Let H be a Sylow p-subgroup of G.

(a) Show that H is abelian and $G = HK$.

(b) Show that there are elements $h \in H$ and $k \in K$ such that $hk \neq kh$, and deduce that p divides $q-1$.

Remark
A Sylow p-subgroup of G is a subgroup whose order is the highest power of p dividing the order of the group G. There always exists at least one Sylow p-subgroup of a group of order $p \cdot m$.

Problem C.1.
Find the limit

$$\lim_{n\to\infty} \left(\int_0^1 e^{x/n} dx \right)^n .$$

Problem C.2.
Calculate the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(n \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) - 1 \right) .$$

Problem E.1.
Prove that

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} dx = \frac{\pi}{4}.$$

Problem E.2.
Find all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) = f'(x)f(y) + f(x)f'(y)$$

for all $x, y \in \mathbb{R}$.

Problem G.1.
Let $q \in (0,1)$ and let us construct a q-spiral as follows. An arc of a quater of circle of radius 1 is drawn inside a $1{\times}1$ square. Then it is connected to a second quater of circle of radius q drawn in a $q{\times}q$ square, and then a third arc is drawn in a $q^2{\times}q^2$, and so on ad infinitum (an example is shown in the figure below). Describe the shape formed by limiting endpoints of all q-spirals for all $q \in (0,1)$.

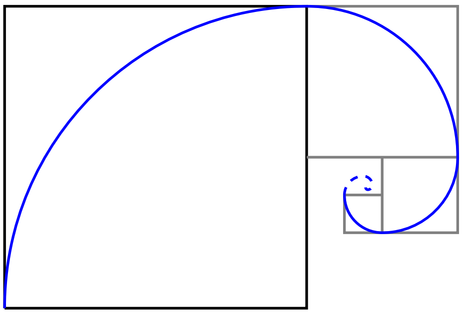

Figure G.1: Example of a q-spiral

Problem G.2.
Let the numbers $x_1, x_2, \ldots x_n$ be in $[0,1]$ and let us consider a matrix A with entries equal to $a_{ij} = |x_i - x_j|$. Find the maximal possible value of $|\det A|$.

Problem P.1.
Does there exist a family of compact sets $A_1, A_2, A_3, \ldots \subset \mathbb{Q}$ such that any compact subset K of rationals is contained in some A_n (i.e. $(K \subset \mathbb{Q}$ and K is compact$)$ $\Rightarrow$ $\exists_n\, K \subseteq A_n$)?
Remark
A subset of real numbers $\mathbb{R}$ is compact if and only if it is closed and bounded in $\mathbb{R}$.

Problem P.2.
Flea Frank is fleeing, jumping along the grid $\mathbb{Z}^2$ from a spider located at origin $(0,0)$. In the first step, the flea jumps from $(0,0)$ to $(1,0)$. In next steps, at each vertex, it jumps to one of the three nearest vertices with equal probability, except for the one it came from. Its choices are random and independent.
Calculate the expected square of the distance of Flea Frank from the origin after n jumps.

SOLUTIONS

Problem A

Let $(R,+,\cdot)$ be a commutative ring. If I and J are two ideals of R then prove that

$$\sqrt{I\cap J}=\sqrt{I}\cap\sqrt{J}$$

and find n such that

$$n\,\mathbb{Z}\;=\;\sqrt{8\,\mathbb{Z}}\cap\sqrt{11\,\mathbb{Z}}\cap\sqrt{2024\,\mathbb{Z}}.$$

Remark

The radical $\sqrt{I}$ of an ideal I is an ideal which consists of all elements in the ring with some power in I, i.e.

$$\sqrt{I}=\{a\in R:\;\underset{n\geqslant 1}{\exists}\;a^n\in I\}\,.$$

[*Proposed by*: Pirmyrat Gurbanov and Murat Chashemov *from*: International University for the Humanities and Development]

Solution:

First, we claim that, if $I\subset J$, then $\sqrt{I}\subset\sqrt{J}$. Indeed, if $x\in\sqrt{I}$, then there exists n such that $x^n\in I\subset J$ and $x\in\sqrt{J}$, from which it follows that $\sqrt{I}\subset\sqrt{J}$.

Since $I\cap J\subset I$ and $I\cap J\subset J$, then $\sqrt{I\cap J}\subset\sqrt{I}$ and $\sqrt{I\cap J}\subset\sqrt{J}$, from which it follows that $\sqrt{I\cap J}\subset\sqrt{I}\cap\sqrt{J}$. Now, suppose $x\in\sqrt{I}\cap\sqrt{J}$. Then, there exist two integers m,n such that $x^n\in I$ and $x^m\in J$. On account of the definition of an ideal, we have $x^n\cdot x^m$ belongs to I and to J.So, $x^n\cdot x^m=x^{n+m}\in I\cap J$. Hence, x is an element of $\sqrt{I\cap J}$ and $\sqrt{I}\cap\sqrt{J}\subset\sqrt{I\cap J}$. From the preceding, we get $\sqrt{I\cap J}=\sqrt{I}\cap\sqrt{J}$, as desired.

As for the second question, in $\mathbb{Z}$ we have that $\sqrt{2024\,\mathbb{Z}}$ is the set of integers x such that there exists a power x^n which is a multiple of 2024. Since $2024=2^3\cdot 11\cdot 23$ then a power of x, say x^n, will be divisible by these factors if and only if x is divisible by $2\cdot 11\cdot 23=506$, and one have that $\sqrt{2024\mathbb{Z}}=506\mathbb{Z}$. Similarly, $\sqrt{11\,\mathbb{Z}}=11\,\mathbb{Z}$ and $\sqrt{8\,\mathbb{Z}}=2\,\mathbb{Z}$.

Finally, we get

$$\sqrt{8\,\mathbb{Z}}\cap\sqrt{11\,\mathbb{Z}}\cap\sqrt{2024\,\mathbb{Z}}=2\,\mathbb{Z}\cap 11\,\mathbb{Z}\cap 506\,\mathbb{Z}=506\,\mathbb{Z}$$

as $506\,\mathbb{Z}\subset 2\,\mathbb{Z}$ and $506\,\mathbb{Z}\subset 11\,\mathbb{Z}$ (both 2 and 11 divide 506).

$\square$

Problem A.1
Find all $n \in \mathbb{N}^+$ such that there exist two cycles c_1, c_2 of length n in S_n, such that c_1 and c_2 are the generators of S_n.
Remark
Generators for a group G is a subset of elements $S \subset G$ such that every element of G can be expressed as a combination (using the group operation) of finitely many elements from S and their inverses. The smallest subgroup of G that contains $\{g_1, \dots, g_n\}$ is denoted $\langle g_1, \dots, g_n \rangle$, and if $\langle g_1, \dots, g_n \rangle = G$, we say that $\{g_1, \dots, g_n\}$ generates G.

[*Proposed by*: Leszek Pieniążek *from*: Jagiellonian University]

Solution:
For odd n cycles c_i are even, so every $\sigma \in \langle c_1, c_2 \rangle$ is even permutation. Thus there are no such cycles.
For $n = 2$ we have obviously positive answer.
Let $n = 2k$, $k > 1$. Let $c_1 = (1, 2, 3, 4, \dots, n)$, $c_2 = (2, 1, 3, 4, \dots, n)$. One can check that $c_3 = \underbrace{c_1 c_2 c_1 \cdots c_2 c_1}_{2k\text{–}1 \text{ times}} = (1, 2)$. Further $c_1 c_3 c_1^{-1} = (2, 3)$, $c_1^2 c_3 c_1^{-2} = (3, 4)$, and so on. The transpositions $(k, k+1)$ for $k = 1, \dots, n-1$ generate S_n, so $\langle c_1, c_2 \rangle = S_n$. □

Problem A.2
Let p and q be distinct prime numbers. Let G be a group of order $p^3 \cdot q$ such that its commutator subgroup K is of order q. Let H be a Sylow p-subgroup of G.

(a) Show that H is abelian and $G = HK$.

(b) Show that there are elements $h \in H$ and $k \in K$ such that $hk \neq kh$, and deduce that p divides $q-1$.

Remark
A Sylow p-subgroup of G is a subgroup whose order is the highest power of p dividing the order of the group G. There always exists at least one Sylow p-subgroup of a group of order $p \cdot m$.

[*Proposed by*: Pirmyrat Gurbanov and Murat Chashemov *from*: International University for the Humanities and Development]

Solution:
(a) Since K is the commutator subgroup of G, K is normal in G and G/K is abelian. So, if $H \cdot K = KH$ is a subgroup of G. Since $|K|= q$, $|H|= p^3$, $H \cap K = \{e\}$ and $|HK|= p^3q = |G|$. Hence $G = HK$ and $H \simeq H/(e) = H/H \cap K \simeq HK/K \simeq G$ is abelian.
(b) Suppose that for any elements $h \in H$ and $k \in K$, we have $hk = kh$. Since K is abelian and H is abelian, then $G = HK$ is abelian. This contradicts the fact that the commutator subgroup K of G is of order q. This proves (b).
(c) First, we claim that H is not normal in G. Otherwise, for any $h \in H$ and $k \in K$, $hkh^{-1}k^{-1} \in H \cap K = \{e\}$ and so $hk = kh$. By Sylow Theorem, the number of Sylow p-subgroups of G is greater than 1 and divides q. So it is q. Again, by Sylow Theorem, $p|q-1$. □

Problem C
Prove that
$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} \notin \mathbb{Q}.$$

[*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Solution:
We put
$$x_k := \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^k}{n!} \text{ for all } k \in \mathbb{N}.$$
It is clear that
$$x_1 = \frac{1}{e} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{e} \cdot e = 1.$$
Now we will show by induction that $x_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Indeed, let us assume that $x_1, ... x_k \in \mathbb{N}$. We will show that $x_{k+1} \in \mathbb{N}$. We have
$$\begin{aligned} e \cdot x_{k+1} &= \sum_{n=1}^{\infty} \frac{n^{k+1}}{n!} = \sum_{n=1}^{\infty} \frac{n^k}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(n+1)^k}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}(n+1)^k \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{k} \binom{k}{m} n^m = \sum_{n=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} \frac{n^m}{n!} = \sum_{m=0}^{k} \sum_{n=0}^{\infty} \binom{k}{m} \frac{n^m}{n!} \\ &= \sum_{m=0}^{k} \binom{k}{m} \sum_{n=0}^{\infty} \frac{n^m}{n!} = e + \sum_{m=1}^{k} \binom{k}{m} \sum_{n=0}^{\infty} \frac{n^m}{n!} = e + \sum_{m=1}^{k} \binom{k}{m} e x_m \\ &= e \left(1 + \sum_{m=1}^{k} \binom{k}{m} x_m \right), \end{aligned}$$
which implies that
$$x_{k+1} = 1 + \sum_{m=1}^{k} \binom{k}{m} x_m \in \mathbb{N}.$$
Thus we have proved that $x_k \in \mathbb{N}$ for all k, and hence
$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} = e \cdot x_{2024} \notin \mathbb{Q}.$$
This completes the solution. □

Solution 2:
Let $f_k(x) = \underbrace{(\ldots((e^x x)' x)' \ldots x)'}_{k \text{ times}}$. Then $f_k(x) = e^x\, P_k(x)$, where P_k is a polynomial of degree k with integer coefficients.
Proof: by induction on k.
We have $f_1(x) = e^x(x+1)$. If $f_k(x) = e^x\, P_k(x)$, then
$$\begin{aligned} f_{k+1}(x) &= (e^x\, P_k(x)\, x)' = e^x\, P_k(x)\, x + e^x\, P'_k(x)\, x + e^x\, P_k(x) \\ &= e^x (P_k(x)\, x + P'_k(x)\, x + P_k(x)), \end{aligned}$$
hence $P_{k+1}(x) = P_k(x)\, x + P'_k(x)\, x + P_k(x)$. ◇
We have
$$f_k(x) = \sum_{n=1}^{\infty} \frac{n^k}{n!} x^{k-1}.$$
by simply differentiation term by term, as all series are absolutely convergent on the whole $\mathbb{R}$.
As P_{2024} has integer coefficients, $P_{2024}(1) \in \mathbb{Z}$. Hence
$$\sum_{n=1}^{\infty} \frac{n^{2024}}{n!} = f_{2024}(1) = e\, P_{2024}(1) \notin \mathbb{Q}.$$
□

Problem C.1
Find the limit
$$\lim_{n\to\infty}\left(\int_0^1 e^{x/n}dx\right)^n.$$

[*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Solution:
Fix $\varepsilon > 0$. We know that
$$e^x \geq 1 + x \text{ for all } x \geq 0$$
and
$$e^x \leq 1 + a_\varepsilon x \text{ for all } x \in [0,\varepsilon] \text{ with } a_\varepsilon = \frac{1}{\varepsilon}(e^\varepsilon - 1) > 0.$$
Observe that
$$\lim_{\varepsilon\to 0^+} a_\varepsilon = 1. \tag{20.1}$$
From the above inequalities we get the following integral estimations:
$$\left(\int_0^1 e^{x/n}dx\right)^n \geq \left(\int_0^1 \left(1+\frac{x}{n}\right)dx\right)^n = \left(1+\frac{1}{2n}\right)^n \xrightarrow[n\to\infty]{} e^{1/2} \tag{20.2}$$
and
$$\left(\int_0^1 e^{x/n}dx\right)^n \leq \left(\int_0^1 \left(1+\frac{a_\varepsilon x}{n}\right)dx\right)^n = \left(1+\frac{a_\varepsilon}{2n}\right)^n \xrightarrow[n\to\infty]{} e^{a_\varepsilon/2} \tag{20.3}$$
Finally, taking into account (20.1), (20.2) and (20.3), we get the conclusion that
$$\lim_{n\to\infty}\left(\int_0^1 e^{x/n}dx\right)^n = e^{1/2}.$$

□

Problem C.2
Calculate the sum
$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)-1\right).$$

[*Proposed by*: Pirmyrat Gurbanov and Murat Chashemov *from*: International University for the Humanities and Development]

Solution:
By Stolz-Cesaro theorem
$$\lim_{N\to\infty}N\sum_{k=N+1}^{\infty}\frac{1}{k^2}=\lim_{N\to\infty}\frac{\zeta(2)-\sum\limits_{k=1}^{N}\frac{1}{k^2}}{\frac{1}{N}}=\lim_{N\to\infty}\frac{\frac{1}{(N+1)^2}}{\frac{1}{N}-\frac{1}{N+1}}=1,$$
so
$$\zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}=\frac{1}{N}+o\left(\frac{1}{N}\right).$$
Hence, as $N\to\infty$
$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)-1\right)\\
&\quad=\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\left(\zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}\right)-1\right)\\
&\quad=-\zeta(2)\sum_{n=1}^{N}(-1)^n n+\sum_{n=1}^{N}(-1)^n n\sum_{k=1}^{n}\frac{1}{k^2}+\sum_{n=1}^{N}(-1)^n\\
&\quad=-\zeta(2)\sum_{n=1}^{N}(-1)^n n+\sum_{n=1}^{N}(-1)^n+\sum_{k=1}^{N}\frac{(-1)^k}{k^2}+\sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=k+1}^{N}(-1)^n n\\
&\qquad+\sum_{k=1}^{N}\frac{1}{k^2}\left(\frac{(-1)^N N-(-1)^k k}{2}+\frac{(-1)^N-(-1)^k}{4}\right)\\
&\quad=-(-1)^N\left(\frac{N}{2}+\frac{1}{4}\right)\left(\zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}\right)+\frac{\zeta(2)}{4}+\frac{(-1)^N-1}{2}\\
&\qquad+\frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^k}{k}-\frac{1}{4}\sum_{k=1}^{N}\frac{(-1)^k}{k^2}\\
&\quad=-(-1)^N\left(\frac{N}{2}+\frac{1}{4}\right)\left(\frac{1}{N}+o\left(\frac{1}{N}\right)\right)+\frac{\zeta(2)}{4}+\frac{(-1)^N-1}{2}\\
&\qquad-\frac{\ln 2}{2}-\frac{\zeta(2)}{8}+o(1)\\
&\quad=\frac{3\zeta(2)}{8}-\frac{\ln 2}{2}-\frac{1}{2}+o(1)\overset{N\to\infty}{\longrightarrow}\frac{\pi^2}{16}-\frac{\ln 2}{2}-\frac{1}{2}.
\end{aligned}$$

□

Solution 2.:
We have
$$1=\int_1^{\infty}\frac{\mathrm{d}x}{x^2}=\sum_{k=n+1}^{\infty}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\frac{\mathrm{d}x}{x^2}=\sum_{k=n+1}^{\infty}\left(-\frac{1}{x}\Bigg|_{\frac{k-1}{n}}^{\frac{k}{n}}\right)=\sum_{k=n+1}^{\infty}\frac{n}{k(k-1)}.$$
Hence
$$n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)-1=n\sum_{k=n+1}^{\infty}\left(\frac{1}{k^2}-\frac{n}{k(k-1)}\right)=-n\sum_{k=n+1}^{\infty}\frac{1}{k^2(k-1)}.$$

Thus

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)-1\right)=\sum_{n=1}^{\infty}(-1)^n n\sum_{k=n+1}^{\infty}\frac{1}{k^2(k-1)}\overset{k\mapsto k+1}{=}\sum_{n=1}^{\infty}(-1)^n n\sum_{k=n}^{\infty}\frac{1}{(k+1)^2k}$$
$$=\sum_{k=1}^{\infty}\frac{1}{(k+1)^2k}\sum_{n=1}^{k}(-1)^n n$$

as we have $\sum_{n=1}^{k}(-1)^n n=-1+2-3+4-\cdots=\begin{cases}m, & k=2m,\\ -m, & k=2m-1,\end{cases}$

$$=\sum_{\substack{k=1\\2|k}}^{\infty}\frac{1}{(k+1)^2k}\,\frac{k}{2}-\sum_{\substack{k=1\\2\nmid k}}^{\infty}\frac{1}{(k+1)^2k}\,\frac{k+1}{2}$$
$$=\sum_{m=1}^{\infty}\frac{1}{2\,(2m+1)^2}-\sum_{m=1}^{\infty}\frac{1}{2\,(2m-1)\,2m}$$

and by renumerating $m\mapsto m-1$ in the first term

$$=\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{(2m-1)^2}-\frac{1}{2}\sum_{m=1}^{\infty}\Big(\frac{1}{2m-1}-\frac{1}{2m}\Big)$$
$$=\frac{1}{2}\Big(\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}-1\Big)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$
$$=\frac{1}{2}\Big(\frac{\pi^2}{6}-\frac{1}{4}\frac{\pi^2}{6}-\frac{1}{2}\Big)-\frac{1}{2}\ln 2=\frac{\pi^2}{16}-\frac{1}{2}\ln 2-\frac{1}{2}\,.$$

□

Problem E
Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(A\triangle B)=f(A)\triangle f(B),$$

where $\triangle$ is the symmetric difference of sets: $A\triangle B=(A\setminus B)\cup(B\setminus A)$.
Remark
We define $f(A)=\{x\in\mathbb{R}\colon \underset{a\in A}{\exists} f(a)=x\}$ *for any subset* $A\subset\mathbb{R}$.

[*Proposed by*: Leszek Pieniążek *from*: Jagiellonian University]

Solution:
One easily checks, that f must be injective (for sets $A=\{a\}$ and $B=\{b\}$, with $a\neq b$). In an obvious way every injection f fulfils the conditions, as

$$f(A\cup B)=f(A)\cup f(B),\qquad f(A\cap B)=f(A)\cap f(B),\qquad f(A\setminus B)=f(A)\setminus f(B)$$

for any sets A and B. $\square$

Problem E.1
Prove that

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} dx = \frac{\pi}{4}.$$

[*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Solution:
Let

$$I := \int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} dx.$$

Now we use the change of variables $y = \frac{\pi}{2} - x$. Thus we get

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} dx = \int_0^{\pi/2} \frac{(\sin y)^{\cos y}}{(\cos y)^{\sin y} + (\sin y)^{\cos y}} dy.$$

Hence

$$2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Longrightarrow I = \frac{\pi}{4}.$$

This completes the solution. □

Problem E.2
Find all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) = f'(x)f(y) + f(x)f'(y)$$

for all $x, y \in \mathbb{R}$.

[*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]

Solution:
It is clear that the constant function $f = 0$ is one of the solutions.
Let us now assume that f is not constantly equal to zero (i.e. $\{x \mid f(x) \neq 0\} \neq \varnothing$). Putting $y = 0$ gives $f(x) = f(x)f'(0) + f'(x)f(0)$, hence either $f(0) = 0$ and $f'(0) = 1$, or

$$\frac{f'(x)}{f(x)} = \frac{1 - f'(0)}{f(0)} = \lambda = \mathrm{const}$$

for all such x that $f(x) \neq 0$. In the second case we have the solution to the differential equation (on the set $\{x \mid f(x) \neq 0\}$) equal to $Ce^{\lambda x}$ for some constant C, which by continuity expands to the whole real line $\mathbb{R}$. Putting this function to the functional equation gives $Ce^{\lambda(x+y)} = 2\lambda C^2 e^{\lambda x} e^{\lambda y}$, hence either $C = 0$ or $\lambda = \frac{1}{2C}$.
Now we return to the first case, i.e. $f(0) = 0$ and $f'(0) = 1$. Hence we have

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = f'(0) = 1\,.$$

Subtracting the functional equations for $y = x_0$ and x or $x + h$ gives

$$f(x + h + x_0) - f(x + x_0) = (f(x+h) - f(x))\, f'(x_0) + (f'(x+h) - f'(x))\, f(x_0)\,,$$

hence

$$\begin{aligned} \frac{f'(x+h) - f'(x)}{h} &= \frac{1}{f(x_0)} \left(\frac{f(x + x_0 + h) - f(x + x_0)}{h} - f'(x_0)\, \frac{f(x+h) - f(x)}{h} \right) \\ &\overset{h\to 0}{\to} \frac{1}{f(x_0)} (f'(x + x_0) - f'(x_0)\, f'(x)) \in \mathbb{R} \end{aligned}$$

when $f(x_0) \neq 0$.
Thus f is twice differentiable and so

$$0 = \frac{\partial f(x+y)}{\partial x} - \frac{\partial f(x+y)}{\partial y} = f''(x)\, f(y) - f(x)\, f''(y)\,,$$

i.e. $f''(x)\, f(y) = f(x)\, f''(y)$ or equivalently $\dfrac{f''(x)}{f(x)} = \lambda = \mathrm{const}$ in a neighbourhood of x_0 (for which $f(x_0) \neq 0$). This gives us either $f''(x) = 0$ (for $\lambda = 0$) or $f''(x) = \lambda\, f(x)$. Hence $f(x) = Cx + D$ in the first case and

$$\begin{aligned} f(x) =& C\, \sin\sqrt{|\lambda|}\, x + D\, \cos\sqrt{|\lambda|}\, x \ \text{ for } \lambda < 0 \\ f(x) =& C\, \sinh\sqrt{\lambda}\, x + D\, \cosh\sqrt{\lambda}\, x \ \text{ for } \lambda > 0 \end{aligned}$$

and those solutions expand by continuity to the whole real line $\mathbb{R}$. Now the conditions $f(0) = 0$ and $f'(0) = 1$ gives $D = 0$ (in both cases) and $C = 1$ in the first case and $C\sqrt{|\lambda|} = 1$ in the second case.
Finally we have that all possible solution are: $0,\ Ce^{\frac{x}{2C}},\ x,\ C\sin\frac{x}{C},\ C\sinh\frac{x}{C}\,.$ □

Problem G
Let x_n denote the maximal determinant of an $n \times n$ matrix with entries equal to ± 1. Does the sequence $\sqrt[n]{x_n}$ have a finite limit?

[*Proposed by*: Robert Skiba *from*: Nicolaus Copernicus University in Toruń]

Solution:
The answer is negative. We will construct a sequence (A_n) of $2^n \times 2^n$ matrices such that

$$\sqrt[2^n]{\det A_n} \xrightarrow[n\to\infty]{} \infty.$$

Indeed, let $A_0 = (1)$ and

$$A_{n+1} = \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix}$$

for all $n \in \mathbb{N}$. Then

$$\det A_{n+1} = \det \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix} = \det \begin{pmatrix} A_n & A_n \\ 0 & 2A_n \end{pmatrix} = 2^{2^n} (\det A_n)^2.$$

One can show by induction that

$$\det A_n = 2^{n2^{n-1}} \text{ for all } n \geq 0.$$

Finally, one has

$$\sqrt[2^n]{\det A_n} = \sqrt[2^n]{2^{n2^{n-1}}} = 2^{n/2} \xrightarrow[n\to\infty]{} \infty.$$

This completes the solution. □

Problem G.1
Let $q \in (0,1)$ and let us construct a q-spiral as follows. An arc of a quater of circle of radius 1 is drawn inside a 1×1 square. Then it is connected to a second quater of circle of radius q drawn in a $q\times q$ square, and then a third arc is drawn in a $q^2\times q^2$, and so on ad infinitum (an example is shown in the figure below). Describe the shape formed by limiting endpoints of all q-spirals for all $q \in (0,1)$.

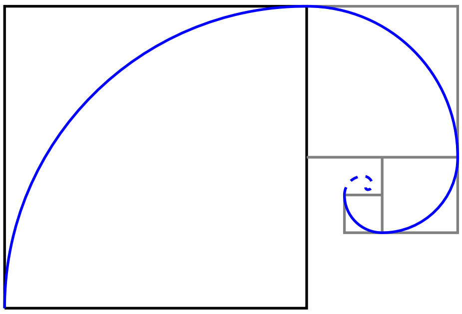

Figure 1: Example of a q-spiral

[*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]

Solution:
We will work in the complex plane $\mathbb{C}$. Let us fix q for a moment and let $z_0 = 0$ be the start point of the spiral. Let then z_n be the point where the spiral exits the n-th square, i.e. $z_1 = i+1$ for example. The diagonals $z_{n-1}z_n$ and $z_n z_{n+1}$ of the consecutive squares meet at right angle, and their lengths differ by a multiple of q, hence

$$z_{n+1} - z_n = (-i)q(z_n - z_{n-1})$$

or equivalently

$$z_{n+1} = (1-iq)z_n + iq\, z_{n-1}$$

for all $n \geq 1$. This is a linear recurrence relation with characteristic equation

$$\lambda^2 = (1-iq)\lambda + iq\,,$$

whose solutions are 1 and $-iq$. So

$$z_n = A\,1^n + B\,(-iq)^n = A + B(-iq)^n$$

for some constants $A, B \in \mathbb{C}$. As $z_0 = 0$ and $z_1 = 1+i$ we get $A = -B = \dfrac{1+i}{1+iq}$, hence

$$z_n = \frac{1+i}{1+iq}(1-(-iq)^n)\,.$$

Thus the limiting endpoint of the spiral is

$$w_q = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{1+i}{1+iq}(1-(-iq)^n) = \frac{1+i}{1+iq} = \frac{1+q+i(1-q)}{1+q^2}\,.$$

The set $\left\{\frac{1+i}{1+iq},\, q\in(0,1)\right\}$ is an arc of a quarter of circle $\left|z - \frac{1+i}{2}\right| = \frac{\sqrt{2}}{2}$ (with center in $\frac{1}{2}(1+i)$ and radius $\frac{\sqrt{2}}{2}$) from 1 to $1+i$ as

$$\left(\frac{1+q}{1+q^2} - \frac{1}{2}\right)^2 + \left(\frac{1-q}{1+q^2} - \frac{1}{2}\right)^2 = \frac{2+4q^2+2q^4}{4(1+q^2)^2} = \frac{1}{2}\,.$$

□

Problem G.2
Let the numbers $x_1, x_2, \dots x_n$ be in $[0,1]$ and let us consider a matrix A with entries equal to $a_{ij} = |x_i - x_j|$. Find the maximal possible value of $|\det A|$.

[*Proposed by*: Leszek Pieniążek *from*: Jagiellonian University]

Solution:
We may assume $0 = x_0 \le x_1 \le x_2 \le \dots \le x_n \le 1$ (permutation of x_i's gives permutation of rows and columns of A which doesn't change the answer). Let $d_i = x_i - x_{i-1} \ge 0$. We have

$$A = \begin{bmatrix} 0 & x_2 - x_1 & x_3 - x_1 & \dots & \dots & x_n - x_1 \\ x_2 - x_1 & 0 & x_3 - x_2 & \dots & \dots & x_n - x_2 \\ x_3 - x_1 & x_3 - x_2 & 0 & x_4 - x_3 & \dots & x_n - x_3 \\ \vdots & \vdots & \vdots & \ddots & & \\ x_n - x_1 & x_n - x_2 & x_n - x_3 & \dots & \dots & 0 \end{bmatrix}.$$

Subtracting from every column the next one doesn't change the determinant and gives the matrix

$$A' = \begin{bmatrix} -d_2 & -d_3 & -d_4 & \dots & \dots & x_n - x_1 \\ d_2 & -d_3 & -d_4 & \dots & \dots & x_n - x_2 \\ d_2 & d_3 & -d_4 & -d_5 & \dots & x_n - x_3 \\ \vdots & \vdots & \vdots & \ddots & & \\ d_2 & d_3 & d_4 & \dots & d_n & 0 \end{bmatrix}.$$

Every column except the last one consist of the same numbers with plus or minus, so we will find determinant of

$$A'' = \begin{bmatrix} -1 & -1 & -1 & \dots & \dots & x_n - x_1 \\ 1 & -1 & -1 & \dots & \dots & x_n - x_2 \\ 1 & 1 & -1 & -1 & \dots & x_n - x_3 \\ \vdots & \vdots & \vdots & \ddots & & \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$

and use $\det A' = \prod_{i=2}^{n} d_i \det A''$. We apply the Laplace expansion due to the last column. Every minor obtained by deleting last column and row number k has two equal columns $k-1$ and k, so we need the determinant of matrix

$$B = \begin{bmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & -1 & \dots & -1 \\ 1 & 1 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

and $\det A'' = (x_n - x_1) \det B (-1)^{n+1}$. If we add the thirst column to every other column we get triangular matrix of dimension $n-1$ with 2's on diagonal in every column except the first one with 1, so $\det B = 2^{n-2}$.
Now

$$|\det A| = |\det A'| = \prod_{i=2}^{n} d_i |\det A''| = (x_n - x_1) \prod_{i=2}^{n} d_i |\det B| = 2^{n-2}(x_n - x_1) \prod_{i=2}^{n} d_i.$$

A-GM inequality gives $\prod_{i=2}^{n} d_i \le \left(\frac{\sum_{i=2}^{n} d_i}{n}\right)^{n-1} = \left(\frac{x_n - x_1}{n-1}\right)^{n-1}$. Moreover $x_n - x_1 \le 1$, so

$$|\det A| \le \frac{2^{n-2}}{(n-1)^{n-1}}$$

with equalities in every $\le$ for $x_i = \frac{i-1}{n-1}$. □

Problem P
Let $(\mathrm{X}, \preccurlyeq)$ be a partially ordered set such that for all subsets $A, B \subset \mathrm{X}$ the following property is satisfied

$$\Big(\underset{\substack{x\in A\\ y\in B}}{\forall} x \preccurlyeq y\Big) \Longrightarrow \underset{z\in \mathrm{X}}{\exists} \Big(\Big(\underset{x\in A}{\forall} x \preccurlyeq z\Big) \wedge \Big(\underset{y\in B}{\forall} z \preccurlyeq y\Big)\Big).$$

(i) Show that every order preserving function $f\colon \mathrm{X} \to \mathrm{X}$ (i.e. $\underset{x,y\in \mathrm{X}}{\forall} x \preccurlyeq y \Rightarrow f(x) \preccurlyeq f(y)$) has a fixed point (i.e. there is an $x_0 \in \mathrm{X}$ such that $f(x_0) = x_0$).

(ii) Give an example of X and f, where the property is satisfied only for non-empty subsets A, B of X and f has no fixed point.

[*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]

Solution:
Ad (i) Let $C = \big\{x\in \mathrm{X}\colon\ x \preccurlyeq f(x)\big\}$ and let $D = \big\{x\in \mathrm{X}\colon \underset{y\in C}{\forall} y \preccurlyeq x\big\}$. By the property there exists $z_o \in \mathrm{X}$ such that $\underset{x\in C}{\forall} x \preccurlyeq z_0$ and $\underset{y\in D}{\forall} z_0 \preccurlyeq y$. Note that if the set C were empty, then D would be equal to X, so by the property z_0 would have to be the smallest element of the set X. In that case, $z_0 \preccurlyeq f(z_0) \in X$, which means that $z_0 \in C$. Hence $C \neq \varnothing$.
We will show that $f(z_0) = z_0$, or more precisely $f(z_0) \preccurlyeq z_0$ and $f(z_0) \succcurlyeq z_0$.
We have $\underset{x\in C}{\forall} x \preccurlyeq z_0 \Rightarrow \underset{x\in C}{\forall} x \preccurlyeq f(x) \preccurlyeq f(z_0)$, hence $f(z_0) \in D$. Moreover, as $\underset{y\in D}{\forall} z_0 \preccurlyeq y$ then $z_0 \preccurlyeq f(z_0)$, hence $z_0 \in C$.
On the other hand applying function f to $x \preccurlyeq f(x)$ gives $f(x) \preccurlyeq f(f(x))$ for such x's, so $f(C) \subset C$ and $f(z_0) \in C$. Thus $f(z_0) \preccurlyeq z_0$.
So by the antisymmetry property of ordering $\preccurlyeq$ we get $f(z_0) = z_0$.

Ad (ii) One can observe that the condition for non-empty sets will always be satisfied whenever each subset of X, which is bounded above (respectively, bounded below), has the largest (respectively, smallest) element. An example of such a set are the integers with the usual order $(\mathbb{Z}, \leq)$. An order-preserving function $f\colon \mathbb{Z} \to \mathbb{Z}$ that does not have a fixed point is, for example, the shift $f(n) = n+1$.

Remark
In the analogical way it can be proved the existence of sup *and* inf *for all subsets in* X. *Then the thesis follows from the well known version of the Banach's Fixpoint Lemma.*

□

Problem P.1
Does there exist a family of compact sets $A_1, A_2, A_3, \ldots \subset \mathbb{Q}$ such that any compact subset K of rationals is contained in some A_n (i.e. $(K \subset \mathbb{Q}$ and K is compact$) \Rightarrow \exists_n K \subseteq A_n$)?
Remark
A subset of real numbers $\mathbb{R}$ is compact if and only if it is closed and bounded in $\mathbb{R}$.

[*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]

Solution:
The answer is negative. We will show it by contradiction.
Assume that such countable family $\{A_1\}_n$ exists. Let $S_n = [\frac{1}{n+1}, \frac{1}{n}] \cap \mathbb{Q}$. We have $S_n \not\subset A_n$ as $\overline{S_n} = [\frac{1}{n+1}, \frac{1}{n}]$ (from the density of $\mathbb{Q}$), . Hence there is a rational number $q_n \in S_n \setminus A_n$. As $\frac{1}{n+1} \leq q_n \leq \frac{1}{n}$ we have that $\lim_{n\to\infty} q_n = 0$, so the set $K = \{0, q_1, q_2, \ldots\}$ is closed. Moreover $K \subset [0, 1]$, so it is compact. But $K \ni q_n \notin A_n$, hence $K \not\subset A_n$ for every n. $\square$

Problem P.2

Flea Frank is fleeing, jumping along the grid $\mathbb{Z}^2$ from a spider located at origin $(0,0)$. In the first step, the flea jumps from $(0,0)$ to $(1,0)$. In next steps, at each vertex, it jumps to one of the three nearest vertices with equal probability, except for the one it came from. Its choices are random and independent.

Calculate the expected square of the distance of Flea Frank from the origin after n jumps.

[*Proposed by*: Marcin J. Zygmunt *from*: University of Silesia, Katowice (Poland)]

Solution:

We will work in the ring $\mathbb{Z}[i]$ instead of the grid $\mathbb{Z}^2$, identifying the pair (n,m) with the number $n+m\,i$.

Let A_n denote the position of the flea after n-th jump. Clearly, the distance of Flea Frank from the origin after n jumps equals to $|A_n|$, so our task is to calculate $\mathbb{E}|A_n|^2$.

We have $A_0 = 0$ and $A_1 = 1$. Moreover the statement of the problem gives the recurrence relation

$$A_{n+1} = A_n + \xi_n(A_n - A_{n-1}),$$

where the random variables ξ_n, $n = 1,2,\dots$ are independent and uniformly distributed over the set $\{1, i, -i\}$ (the flea with equal probability either continues in the previous direction, turns left – which corresponds to multiplication by i –, or turns right – corresponding to multiplication by $-i$). We have $\mathbb{E}(\xi_n) = \mathbb{E}\left(\overline{\xi_n}\right) = \frac{1}{3}$ and $|\xi_n| = |\xi_n|^2 = 1$.

Now setting $B_{n+1} = A_{n+1} - A_n$, we get

$$B_{n+1} = \xi_n B_n = \cdots = \xi_n \cdots \xi_1 B_1 = \prod_{k=1}^{n} \xi_k \,.$$

Hence,

$$A_n = B_n + A_{n-1} = \cdots = B_n + \cdots + B_1 + A_1 = 1 + \sum_{m=1}^{n-1} \prod_{k=1}^{m} \xi_k \,,$$

and

$$\begin{aligned}
|A_n|^2 = A_n\,\overline{A_n} &= \left(1 + \sum_{m=1}^{n-1}\prod_{k=1}^{m}\xi_k\right)\left(1 + \sum_{m=1}^{n-1}\prod_{k=1}^{m}\overline{\xi_k}\right) \\
&= 1 + \sum_{m=1}^{n-1}\prod_{k=1}^{m}|\xi_k|^2 + \sum_{m=1}^{n-1}\left(\prod_{k=1}^{m}\xi_k + \prod_{k=1}^{m}\overline{\xi_k}\right) + \sum_{m=1}^{n-2}\sum_{j=m+1}^{n-1}\left(\prod_{k=1}^{m}\xi_k\prod_{l=1}^{j}\overline{\xi_l} + \prod_{k=1}^{m}\overline{\xi_k}\prod_{l=1}^{j}\xi_l\right) \\
&= 1 + \sum_{m=1}^{n-1}\prod_{k=1}^{m}1 + \sum_{m=1}^{n-1}\left(\prod_{k=1}^{m}\xi_k + \prod_{k=1}^{m}\overline{\xi_k}\right) + \sum_{m=1}^{n-2}\sum_{j=m+1}^{n-1}\prod_{k=1}^{m}|\xi_k|^2\left(\prod_{l=m+1}^{j}\xi_l + \prod_{l=m+1}^{j}\overline{\xi_l}\right) \\
&= n + \sum_{m=1}^{n-1}\left(\prod_{k=1}^{m}\xi_k + \prod_{k=1}^{m}\overline{\xi_k}\right) + \sum_{m=1}^{n-2}\sum_{j=m+1}^{n-1}\left(\prod_{l=m+1}^{j}\xi_l + \prod_{l=m+1}^{j}\overline{\xi_l}\right).
\end{aligned}$$

Using the linearity of expectation and the independence of the variables ξ_n, we finally obtain that

$$\begin{aligned}
\mathbb{E}|A_n|^2 &= n + 2\sum_{m=1}^{n-1}\left(\frac{1}{3}\right)^m + 2\sum_{m=1}^{n-2}\sum_{j=m+1}^{n-1}\left(\frac{1}{3}\right)^{j-m} = n + \frac{3^{n-1}-1}{3^{n-1}} + 2\sum_{l=1}^{n-2}\frac{n-l-1}{3^l} \\
&= 2n + \frac{3}{2}\left(\frac{1}{3^n} - 1\right).
\end{aligned}$$

□