ISTOCiM 2020 Problems

booklet with solutions

Problem A.1

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the ring of integers modulo n. Find all pairs of non-empty subsets A and B such that $A \cup B = \mathbb{Z}_n$, $A \cap B = \varnothing$, $A + B \subset A$ and $A \cdot B \subset B$.
(We define $A + B = \{a + b \;:\; a \in A,\, b \in B\}$ and $A \cdot B = \{a \cdot b \;:\; a \in A,\, b \in B\}$, with all operations performed modulo n).

Solution:

Let us consider any partition of $\mathbb{Z}_n$ that meets the conditions. Note that if $1 \in B$, then for any $a \in A$ we have $a = a \cdot 1 \in B$, which contradicts the assumption that sets A and B are distinct. Therefore $1 \in A$. In the similar way, as $a \cdot 0 = 0$, we show that $0 \in B$.
Let now consider the set of integers $\mathbb{Z}$ as a covering set for $\mathbb{Z}_n$ equipped with classical retraction $\psi : \mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/_{n\mathbb{Z}}$. The counterimages $\hat{A} = \psi^{-1}(A) = \{a + kn : a \in A, k \in \mathbb{Z}\}$ and $\hat{B} = \psi^{-1}(B) = \{b + kn : b \in B, k \in \mathbb{Z}\}$ satisfy the same conditions as A and B, now in the ring $\mathbb{Z}$ (i.e. $\hat{A} \cup \hat{B} = \mathbb{Z}_n$, $\hat{A} \cap \hat{B} = \varnothing$, $\hat{A} + \hat{B} \subset \hat{A}$ and $\hat{A} \cdot \hat{B} \subset \hat{B}$).
We have $a + (-a) = 0 \in \hat{B}$, hence either both a and $-a$ belongs to $\hat{A}$, or both to $\hat{B}$.
Let now d be the smallest positive element of $\hat{B}$. Hence $1, 2, \ldots, d-1 \in A$. It is easy to prove inductively that $\{1, 2, \ldots, d-1\} + \{d, 2d, \ldots, kd\} \subset \hat{A}$ for each integer k, which leads us to the conclusion that all numbers not divisible by d belong to the set $\hat{A}$. In particular, it follows that each number divisible by d, but not divisible by d^2, belongs to the set $\hat{B}$
Suppose now that $kd^2 \in \hat{A}$ for some integer of k. Then, as $d \in \hat{B}$, we have $kd^2 + d = (kd+1)d \in \hat{A}$, which contradicts the conclusion of the previous paragraph. This means that all numbers divisible by d belong to the set $\hat{B}$, and those not divisible by d – to the set $\hat{A}$.
Finaly, as $d + n \in \hat{B}$ (i.e. $d + n = kd$ for some integer k), the number d is a divisor of n.
It remains to be noted that, if $B \subset \mathbb{Z}_n$ is the subset of numbers divisible by a fixed divisor of n, and A is the subset of numbers indivisible by it, the conditions are met. $\square$

Problem A.2
Find all polynomials $P\colon \mathbb{R}\to\mathbb{R}$ such that $P(\mathbb{Q})=\mathbb{Q}$.

Solution:
Suppose that $P(x)=\sum_{j=0}^{n} a_j x^j$ for some $a_0,\ldots,a_n\in\mathbb{R}$ and $n\in\mathbb{N}$ with $a_n\neq 0$ and every $x\in\mathbb{R}$. Then the matrix

$$A=\begin{bmatrix} 1 & 1 & \cdots & 1\\ 1 & 2 & \cdots & 2^n\\ \vdots & \vdots & \ddots & \vdots\\ 1 & n+1 & \cdots & (n+1)^n \end{bmatrix}\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix}=\begin{pmatrix} P(1)\\ P(2)\\ \vdots\\ P(n+1)\end{pmatrix}$$

is a Vandermonde matrix and $\det(A)\neq 0$. Let $A^{-1}=[b_{i,j}]_{1\leqslant i,j\leqslant n+1}$. Then

$$b_{i,j}=\frac{(-1)^{i+j}\det(A \text{ with dropped } i\text{-row and } j\text{-column})}{\det(A)}$$

for all $1\leqslant i,j\leqslant n+1$. Since determinants of matrices with rational entries are rational numbers, $b_{i,j}$ is a rational number for all $1\leqslant i,j\leqslant n+1$. Consequently

$$\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix}=A^{-1}\begin{pmatrix} P(1)\\ P(2)\\ \vdots\\ P(n+1)\end{pmatrix}$$

and $a_0,\ldots,a_n\in\mathbb{Q}$. Hence there exists $m\in\mathbb{N}$ such that $ma_j\in\mathbb{Z}$ for every $0\leqslant j\leqslant n$.
Suppose now that $n>1$. Let s be a prime number such that $s>|ma_n|+\cdots+|ma_0|$. Suppose that there exists relatively prime numbers $p\in\mathbb{Z}$ and $q\in\mathbb{N}$ such that

$$m\,P\left(\frac{p}{q}\right)=s+ma_0.$$

Then

$$ma_np^n+\ldots+ma_1pq^{n-1}=p(ma_np^{n-1}+\ldots+ma_1q^{n-1})=q^ns$$

and $p|s$ and $q|ma_n$. Since $s>|ma_n|+\ldots+|ma_0|$ and s is a prime number, $|p|>q$ and $p=\pm s$. Then

$$|ma_ns^n|=|ma_np^n|=q|q^{n-1}s-ma_{n-1}p^{n-1}-\ldots-ma_1q^{n-2}p|<q^ns+q(s-|ma_n|)s^{n-1}.$$

This is equivalent to

$$|ma_n|s^n+|ma_n|s^{n-1}q<q^ns+qs^n.$$

But $|ma_n|\geqslant q$. We have arrived at a contradiction.
Therefore $n=1$ and $P(x)=a_1x+a_0$ for some $a_0,a_1\in\mathbb{Q}$. Now it is easy to see that $P(\mathbb{Q})=\mathbb{Q}$, if and only if $P(x)=a_1x+a_0$ and $a_0,a_1\in\mathbb{Q}$ and $a_1\neq 0$. □

Problem C.1

Suppose that $f\colon [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and differentiable on $(0,1)$ with $f(0) = 0$ and $f(1) = 1$. Prove that there exist 2020 distinct points $x_k \in (0,1)$ such that

$$\sum_{k=1}^{2020} \frac{1}{f'(x_k)} = 2020.$$

Solution:

Let us choose 2021 points

$$w_0 = 0 < w_1 < w_2 < \cdots < w_{2019} < w_{2020} = 1$$

with the property:

$$f(w_k) = \frac{k}{2020} \quad \text{for } 0 \leqslant k \leqslant 2020.$$

This can be done since f admits the Darboux property. Now let us observe that the Mean value theorem implies the existence of points $x_k \in (w_{k-1}, w_k)$ such that

$$\frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} = f'(x_k).$$

But

$$\frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} = \frac{k/2020 - (k-1)/2020}{w_k - w_{k-1}} = \frac{1}{2020} \frac{1}{w_k - w_{k-1}}.$$

Consequently, one has

$$\sum_{k=1}^{2020} \frac{1}{f'(x_k)} = \sum_{k=1}^{2020} 2020(w_k - w_{k-1}) = 2020 \sum_{k=1}^{2020} (w_k - w_{k-1}) = 2020(w_{2020} - w_0) = 2020.$$

$\square$

Problem C.2
Let $f\colon [0,1] \to [A,B]$ be a measurable function, where $A < B$ are positive numbers. Prove that

$$\int\limits_0^1 f(x)\,dx - \left(\int\limits_0^1 \frac{dx}{f(x)}\right)^{-1} \leqslant \left(\sqrt{B}-\sqrt{A}\right)^2,$$

and find a function for which equality is achieved.

Solution:
Let $s(x)$ be defined by the relation $f(x) = s(x)A + (1-s(x))B$ for all $x \in [0,1]$, and let $t = \int\limits_0^1 s(x)dx$. We have $\int\limits_0^1 f(x)dx = tA + (1-t)B$.
On the other hand, by convexity of homographic function $\frac{1}{x}$, we have

$$\frac{1}{f(x)} \leqslant \frac{s(x)}{A} + \frac{1-s(x)}{B},$$

hence

$$\int\limits_0^1 \frac{dx}{f(x)} \leqslant \frac{t}{A} + \frac{1-t}{B} = \frac{tB+(1-t)A}{AB}.$$

Also note that

$$0 \leqslant (t\sqrt{a} - (1-t)\sqrt{B})^2 = tB + (1-t)A - t(1-t)\left(\sqrt{B}+\sqrt{A}\right)^2,$$

so

$$tB + (1-t)A \geqslant t(1-t)\left(\sqrt{B}+\sqrt{A}\right)^2.$$

Now

$$\int\limits_0^1 f(x)\,dx - \left(\int\limits_0^1 \frac{dx}{f(x)}\right)^{-1} \leqslant tA + (1-t)B - \frac{AB}{tB+(1-t)A} = \frac{t(1-t)(B-A)^2}{tB+(1-t)A}$$

$$\leqslant \frac{t(1-t)(B-A)^2}{t(1-t)\left(\sqrt{B}+\sqrt{A}\right)^2} = \left(\sqrt{B}-\sqrt{A}\right)^2,$$

as claimed.

To show the equality let us take function

$$f(x) = \begin{cases} A, & 0 \leqslant x \leqslant \frac{\sqrt{A}}{\sqrt{A}+\sqrt{B}}, \\ B, & \frac{\sqrt{A}}{\sqrt{A}+\sqrt{B}} < x \leqslant 1. \end{cases}$$

Then

$$\int\limits_0^1 f(x)\,dx = \frac{A\sqrt{A}+B\sqrt{B}}{\sqrt{A}+\sqrt{B}},$$

and

$$\int\limits_0^1 \frac{dx}{f(x)} = \frac{1}{\sqrt{A}\sqrt{B}}.$$

Hence

$$\int\limits_0^1 f(x)\,dx - \left(\int\limits_0^1 \frac{dx}{f(x)}\right)^{-1} = \frac{A\sqrt{A}+B\sqrt{B}}{\sqrt{A}+\sqrt{B}} - \sqrt{A}\sqrt{B} = \frac{(B-A)(\sqrt{B}-\sqrt{A})}{\sqrt{A}+\sqrt{B}} = \left(\sqrt{B}-\sqrt{A}\right)^2.$$

□

Problem E.1
Show that both equations (1) $\sin(\tan(x)) = x$ and (2) $\tan(\sin(x)) = x$ have a unique positive solution. Determine which one is greater.

Solution:
We start with the following lemma:

Lemma 5.1
The equation

$$\sin x = \arctan x \tag{5.1}$$

has a unique positive solution x_0. *It satisfies* $1 < x_0 < \sqrt{3}$.

Proof: Let $f(x) = \sin x - \arctan x$. It is enough to show that the function f has only one positive zero, and that it is located in the interval $(1, \sqrt{3})$.
We have $f(0) = 0$ and

$$f'(x) = \cos x - \frac{1}{1+x^2} > 1 - \frac{1}{2}x^2 - \frac{1}{1+x^2} = \frac{x^2(1-x^2)}{2(1+x^2)} > 0,$$

for $0 < x < 1$. Hence $f(x) > 0$ for $0 < x \leqslant 1$.
On the other hand we have $\arctan x \geqslant \arctan\sqrt{3} = \frac{\pi}{2} > 1 \geqslant \sin x$, that is $f(x) < 0$, for $x \geqslant \sqrt{3}$.
Hence, due to the continuity, the function f has at least one positive zero, and all its positive zeros lie in the interval $(1, \sqrt{3})$.
Now we have

$$f''(x) = -\sin x + \frac{2x}{(1+x^2)^2} < -\sin\frac{\pi}{4} + \frac{2x}{(2x)^2} \leqslant -\frac{\sqrt{2}}{2} + \frac{1}{2} < 0$$

for $1 \leqslant x \leqslant \sqrt{3}$. Hence f is a concave function on the interval $[1, \sqrt{3}]$, so, due to $f(1) > 0 > f(\sqrt{3})$, it has exactly one zero in this interval.

◇

Let x_0 be the positive solution of (5.1) (as in lemma 5.1). Since $\sin x \leqslant 1 < \frac{\pi}{2}$, we can apply a tangent to both sides of the equation. Hence the equation $\tan(\sin x) = x$ has the same solution as (5.1), i.e. x_0.
On the other hand, substituting $tan(x)$ for x in (5.1) gives the first of the considered equations. So it has a unique solution $x_1 = \arctan x_0$ in the interval $\left(0, \frac{\pi}{2}\right)$. But left-hand side of the equation does not exceed 1, so there are no more other positive solutions to the equation for $x > \frac{\pi}{2}$. Moreover we have also $x_1 \leqslant 1 < x_0$. □

Problem E.2
Find all functions $f\colon [0,\infty)\to[0,\infty)$ such that $2f(3x)+4f(3y)\leqslant 3f(2x+4y)$ for all $x,y\geqslant 0$

Solution:
We claim that the only solutions are functions of the form $f(x)=ax$ with $a\geqslant 0$.
First, let $F(x)=f(3x)$. Then the inequality takes form

$$\frac{2}{3}F(x)+\frac{4}{3}F(y)\leqslant F\Big(\frac{2}{3}x+\frac{4}{3}y\Big). \tag{6.1}$$

Now putting $x,y=0$ gives $2F(0)\leqslant F(0)$, so we have $F(0)=0$. We will show that if $F(c)=0$ for some $c>0$, then $F(x)=0$ for all x. Indeed, if x and y satisfy $\frac{2}{3}x+\frac{4}{3}y=c$, then $\frac{2}{3}F(x)+\frac{4}{3}F(y)\leqslant 0$, and so $F(x)=F(y)=0$. It follows that $F(x)=0$ for all x in the interval $\Big[0,\frac{3}{2}c\Big]$, and then for all x in the interval $\Big[0,\big(\frac{3}{2}\big)^2c\Big]$, and, continuing in this way, finally $F(x)=0$ for all $x\geqslant 0$. In this case, $F(x)=0=0\cdot x$ for all x.
Now assume $F(x)>0$ for all $x>0$. Two special cases of our condition (second variable set to zero) are $\frac{2}{3}F(x)\leqslant F\big(\frac{2}{3}x\big)$ and $\frac{4}{3}F(x)\leqslant F\big(\frac{4}{3}x\big)$. These are equivalent to

$$\frac{F(x)}{x}\leqslant\frac{F\big(\frac{2}{3}x\big)}{\frac{2}{3}x}\ \text{ and }\ \frac{F(x)}{x}\leqslant\frac{F\big(\frac{4}{3}x\big)}{\frac{4}{3}x}\ \text{ for } x>0.$$

Let $A=\inf\Big\{\frac{F(x)}{x}:1\leqslant x\leqslant\frac{4}{3}\Big\}$. Using the second inequality once again leads to $A=\inf\Big\{\frac{F(x)}{x}:1\leqslant x\leqslant\big(\frac{4}{3}\big)^2\Big\}$, and, continuing in this way, finally gives $A=\inf\Big\{\frac{F(x)}{x}:1\leqslant x<\infty\Big\}$. We then use the first inequality in the same way to get $A=\inf\Big\{\frac{F(x)}{x}:0<x<\infty\Big\}$.
Next let $G(x)=F(x)-Ax$. The function G is non negative and satisfying (6.1). Moreover $\inf\Big\{\frac{G(x)}{x}:1\leqslant x\leqslant\frac{4}{3}\Big\}=0$. We claim $G(1)=0$. Indeed, $G\big(\frac{2}{3}+\frac{4}{3}y\big)\geqslant\frac{2}{3}G(1)+\frac{4}{3}G(y)\geqslant\frac{2}{3}G(1)$ for all y. So $G(x)\geqslant\frac{2}{3}G(1)$ for all $x\geqslant\frac{2}{3}$. Let now $\varepsilon>0$ and $y\in\big[1,\frac{4}{3}\big]$ with $g(y)<\varepsilon$. Since $y\geqslant 1>\frac{2}{3}$, we have $AG(1)\leqslant G(y)<\varepsilon$. Since ε is arbitrary, we conclude that $G(1)=0$. Thus $G(x)=0$ for all $x\geqslant 0$, which yields $F(x)=Ax$, and $f(x)=A\frac{x}{3}=ax$ for all $x\geqslant 0$. $\square$

Problem G.1
Let P_1, P_2, P_3 be points on a parabola, and denote the triangle formed by the tangents to the parabola at these points as $\triangle Q_1Q_2Q_3$. Compute the ratio of the area of the triangle $\triangle P_1P_2P_3$ to the area of the triangle $\triangle Q_1Q_2Q_3$.

Solution:
Let us choose a Cartesian system of coordinates such that the equation of the parabola is $x^2 = 4ay$, and let the coordinates of the points be $P_i = (4ax_i, 4ax_i^2)$, for appropriately chosen x_i, $i = 1, 2, 3$. Hence the three tangent lines (denoted l_i respectively) are given by equations $2x_ix = y + 4ax_i^2$, for $i = 1, 2, 3$. Then Q_k as the intersection of l_i and l_j is $(2a(x_i + x_j), 4ax_ix_j)$. The area of both triangles can be computed by a Vandermonde determinant:

$$S_{\triangle P_1P_2P_3} = \frac{1}{2} \cdot \left| \det \begin{bmatrix} 4ax_1 & 4ax_1^2 & 1 \\ 4ax_2 & 4ax_2^2 & 1 \\ 4ax_3 & 4ax_3^2 & 1 \end{bmatrix} \right| = 8a^2 \left| (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \right|$$

and

$$S_{\triangle Q_1Q_2Q_3} = \frac{1}{2} \cdot \left| \det \begin{bmatrix} 2a(x_2 + x_3) & 4ax_2x_3 & 1 \\ 2a(x_1 + x_3) & 4ax_1x_3 & 1 \\ 2a(x_1 + x_2) & 4ax_1x_2 & 1 \end{bmatrix} \right|$$

$$= 4a^2 \left| \det \begin{bmatrix} 2a(x_3 - x_1) & 4ax_2(x_3 - x_1) & 0 \\ 2a(x_3 - x_2) & 4ax_1(x_3 - x_2) & 0 \\ 2a(x_1 + x_2) & 4ax_1x_2 & 1 \end{bmatrix} \right| = 4a^2 \left| (x_3 - x_1)(x_2 - x_3)(x_1 - x_3) \right|.$$

We conclude that the ratio of the two areas is 2, regardless of the location of the three points or the shape of the parabola. $\square$

Problem G.2
A disk of radius R is covered by m rectangular strips of width 2 and infinite length. Prove that $m \geqslant R$.

Solution:
Let us move to three dimensions. The disk is a projection of a sphere of radius R to the xy-plane. And the strips are images of a space contained between two parallel planes, both perpendicular to xy-plane. Now the problem becomes easy. The argument is based on the following property of the sphere.

Lemma 8.1
The area of the surface cut from a sphere of radius R by two parallel planes at distance d from each other is equal to $2\pi Rd$.

Proof: Let us assume that the sphere is centered at the origin and the planes are perpendicular to the x-axis, namely they are $\{x = a\}$ and $\{x = b\}$. The surface is obtained by rotating the graph of the function $f : [a, b] \to \mathbb{R}$, $f(x) = \sqrt{R^2 - x^2}$ about the x-axis, where $[a, b]$ is an interval of length d. The area of the surface is given by integral

$$2\pi \int_a^b f(x) \sqrt{\left(f'(x)\right)^2 + 1}\, dx = 2\pi \int_a^b R\, dx = 2\pi Rd\,.$$

$\diamond$

Returning to the problem, the sphere has area $4\pi R^2$ and is covered by m surfaces, each – by the lemma 8.1 – having area $4\pi R$. The inequality $m \cdot 4\pi R \geqslant 4\pi R^2$ implies $m \geqslant R$, as desired.
$\square$

Problem P.1

A group of k people play a game with a box containing k balls with their names. The first player draws the ball and wins if his/her name is on the ball. Otherwise, the ball is returned to the box, and the person whose name was on the drawn ball is the next player to draw. The procedure continues until someone pulls out the ball with his/her name on it.
What is the probability of winning for each player?

Solution:
Let $A_i = \{\text{player } i \text{ wins}\}$ and $F_i = \{\text{first ball containes the name of player } i\}$. Then, using conditional probabilities, we can write for every $i \neq 1$ (each player with $i \neq 1$ plays in the same conditions)

$$P(A_i) = P(A_i|F_1)P(F_1) + P(A_i|F_i)P(F_i) + \sum_{1 \neq j \neq i} P(A_i|F_j)P(F_j).$$

But $P(A_i|F_1) = 0$ (first player wins in the first draw), $P(A_i|F_i) = P(A_1)$ (after first draw player i is in the role of beginning player) and $P(A_i|F_j) = P(A_i)$. Thus above equation can be written as

$$P(A_i) = P(A_1)\frac{1}{k} + (k-2)P(A_i)\frac{1}{k} \iff P(A_1) = 2P(A_i).$$

Moreover probability of infinite game is 0 (as the probability that the game will last longer then m turns is $(\frac{k-1}{k})^m \to 0$ as $m \to \infty$), so we get $P(A_1) = \frac{2}{k+1}$ and $P(A_i) = \frac{1}{k+1}$ for $i \neq 1$. $\square$

Problem P.2

Let $n \in \mathbb{N}$, $n > 1$ and $c \geqslant 1$. Suppose that $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are two families of independent events in a probability space (Ω, Σ, P) such that $B_k \subset A_k$ and $c \cdot P(B_k) \geqslant P(A_k)$ for every $k = 1, \dots, n$. Show that

$$c \cdot P(B_1 \cup \dots \cup B_n) \geqslant P(A_1 \cup \dots \cup A_n).$$

Solution:

It is clear that we may assume that $c > 1$. Suppose first that $n = 2$. It is easy to check that $p + q - pq \geqslant 0$ for all $p, q \in [0, 1]$. Let $d_j = \frac{cP(B_j) - P(A_j)}{(c-1)P(A_j)}$ for every $j = 1, 2$. Then $0 \leqslant d_j \leqslant 1$ and $0 \leqslant P(A_j)d_j \leqslant 1$ for every $j = 1, 2$. We have

$$\begin{aligned} & c \cdot P(B_1 \cup B_2) - P(A_1 \cup A_2) \\ &= c\big(P(B_1) + P(B_2) - P(B_1)P(B_2)\big) - P(A_1) - P(A_2) + P(A_1)P(A_2) \\ &= (c-1)\big(P(A_1)d_1 + P(A_2)d_2 - P(A_1)P(A_2)d_1d_2\big) \\ &+ \frac{(cP(B_1) - P(A_1))(cP(B_2) - P(A_2)) - c(c-1)P(B_1)P(B_2) + (c-1)P(A_1)P(A_2)}{c-1} \\ &\geqslant \frac{(cP(B_1)P(B_2) - cP(B_1)P(A_2) - cP(B_2)P(A_1) + cP(A_1)P(A_2)}{c-1} \\ &= \frac{c(P(A_1) - P(B_1))(P(A_2) - P(B_2))}{c-1} \geqslant 0. \end{aligned}$$

Let now $n > 2$. We apply $n-1$ times the above procedure with the fact that family of events $A_1 \cup A_2, A_3, \dots, A_n$ as well as $B_1 \cup B_2, B_3, \dots, B_n$ are independent. $\square$