PROBLEMS for individual part

Problem A (Algebra & Combinatorics) [*Proposed by* Artur Michalak *from* Adam Mickiewicz University, Poznań]
Let $n, m \in \mathbb{N}$, $m \leqslant n$. Find the value of the sum

$$\sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \binom{n}{k}.$$

Solution to this problem is on page no. 4

Problem C (Calculus & Mathematical Analysis) [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Does there exist a sequence (a_n) of positive real numbers such that $\lim\limits_{n\to\infty} a_n = 0$, while the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+a_n}}$ converges? If so, provide an example.

Solution to this problem is on page no. 5

Problem E (Equations & Inequalities) [*Proposed by* Robert Skiba *from* Nicolaus Copernicus University in Toruń (Poland)]
For any positive integer n, find all nonnegative integers (x, y) such that

$$\frac{x! + y!}{n!} = 3^{n!}.$$

Solution to this problem is on page no. 6

Problem G (Geometry & Linear Algebra) [*Proposed by* Pirmyrat Gurbanov & Murat Chashemov *from* International University for the Humanities and Development (Turkmenistan)]
Let $A, B \in M_n(\mathbb{R})$ be symmetric matrices such that

$$(AB + BA - A - B - I_n)^2 = A^2B - 2ABA + BA^2.$$

Find $\operatorname{rank}(A^2 + B^2)$.

Solution to this problem is on page no. 8

Problem P (Probability & Set Theory) [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Let n be a fixed positive integer. The random experiment involves repeatedly rolling a fair die and recording the subsequent results until we get n consecutive ones. Calculate the expected value of the total sum of points obtained. We assume the stochastic independence of rolls.

Solution to this problem is on page no. 9

Excerpt from the Rules

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&7 Problems

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7. Solutions are evaluated (both in the individual and team part) in the following scale: $0, 3, 8, 10$ points, where:

a) 10 points are awarded for a complete solution (even with minor errors);

b) 8 points are awarded for a solution that is basically correct but contains major faults (e.g. a calculation error that simplified the reasoning, lack of substantial justification, etc.);

c) 3 points are awarded for a solution that is incomplete but contains a major step towards a correct solution;

d) 0 points are awarded to every other solution, even partial.

8. Solutions should be formulated in a clear, precise and legible manner that excludes ambiguity; unclear, imprecise or difficult to read solutions may result in a reduction in points, down to 0 points.

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SOLUTIONS

Solution to the Problem A:

$$\sum_{k=m}^{n}(-1)^{k-m}\binom{k}{m}\binom{n}{k} = \sum_{k=m}^{n}(-1)^{k-m}\frac{n!}{m!\,(k-m)!}\,\frac{n!}{n!\,(n-k)!} =$$

$$= \begin{cases} 1 & \text{if } n = m \\ \frac{\left(\prod_{k=n-m+1}^{n} k\right)\left(\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}\right)}{m!} = \frac{\left(\prod_{k=n-m+1}^{n} k\right)(1-1)^{n-m}}{m!} = 0 & \text{if } k > m. \end{cases}$$

□

Solution to the Problem C:
Yes, such a sequence exists. There are many simple examples of such sequences.
7 The convergence of the series does not depend on a finite number of first terms of its initial terms, so we can define the sequence (a_n) as $a_1 = a_2 = 1$ and $a_n = \dfrac{2\ \ln\ln n}{\ln n} = 2\log_n(\ln n)$ for $n \geqslant 3$. So we have

$$n^{1+a_n} = n^{1+2\log_n \ln n} = n\ \ln^2 n\,.$$

By Cauchy's Condensation Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\ \ln^2 n}$ is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} 2^n \frac{1}{2^n\ \ln^2(2^n)} = \sum_{n=1}^{\infty} \frac{1}{n^2\ \ln^2 2}\,,$$

so the series converges. □

Solution to the Problem E:
Fix n and suppose (x, y) is a solution. Without loss of generality, assume

$$x \leq y.$$

Case 1: $x \leq n$**.** Equation (1) becomes

$$1 + \frac{y!}{x!} = \frac{3^{n!} \cdot n!}{x!}. \tag{2}$$

This implies

$$1 + \frac{y!}{x!} \equiv 0 \pmod 3.$$

Hence $x < y$, and moreover the factor $\frac{y!}{x!} = (x+1)(x+2)\cdots y$ is not divisible by 3. Thus in the interval $[x+1, y]$ no multiple of 3 can occur, forcing $y \leq x + 2$.
So there are two possibilities:

1. $y = x + 1$,
2. $y = x + 2$.

Subcase (a): $y = x + 2$. Then (2) gives

$$1 + (x+1)(x+2) = \frac{3^{n!}\, n!}{x!}. \tag{3}$$

The left-hand side is odd, so the right-hand side must also be odd. Hence $n \leq x + 1$.
– If $n = x$, then (3) becomes

$$x^2 + 3x + 3 = 3^{x!}.$$

For $x \geq 1$, the left-hand side $\equiv 3 \pmod 9$, whereas $3^{x!}$ is divisible by 9 when $x! \geq 2$. Contradiction.
– If $n = x + 1$, then (3) gives

$$1 + (x+1)(x+2) = 3^{(x+1)!}(x+1).$$

This forces $x + 1 \mid 1$, hence $x = 0$, so $(x, y, n) = (0, 2, 1)$. By symmetry also $(2, 0, 1)$ is a solution.

Subcase (b): $y = x + 1$. Then (2) gives

$$x + 2 = \frac{3^{n!}\, n!}{x!}. \tag{4}$$

If $n \neq x$, then the right-hand side is divisible by $x + 1$, while the left-hand side is congruent to 1 modulo $x + 1$. Impossible. Thus $n = x$ and we get

$$x + 2 = 3^{x!}.$$

The unique solution is $x = 1$, giving $(1, 2, 1)$, and by symmetry $(2, 1, 1)$.

So in Case 1 we obtain four solutions:

$$(0, 2, 1),\ (2, 0, 1),\ (1, 2, 1),\ (2, 1, 1).$$

Case 2: $x > n$**.**
Equation (1) becomes

$$\frac{x!}{n!} + \frac{y!}{n!} = 3^{n!}. \tag{5}$$

If $x \geq n + 2$, then both $x!/n!$ and $y!/n!$ contain factors $(n+1)(n+2)$, so the left-hand side is divisible by both $n + 1$ and $n + 2$, while the right-hand side is a pure power of 3. This is impossible, hence we must have $x = n + 1$.
Thus (5) becomes

$$n + 1 + \frac{y!}{n!} = 3^{n!}. \tag{6}$$

Write $M = \frac{y!}{(n+1)!}$. Then

$$(n+1)((n+1)M + 1) = 3^{n!}. \tag{7}$$

If $y \geq n + 4$, then M is divisible by 3, hence $(n+1)M + 1$ is not a power of 3. Contradiction. So $y \leq n + 3$, giving three subcases.

- $y = n + 1$: Then $M = 1$, and (7) gives $(n+1)(n+2) = 3^{n!}$, impossible.
- $y = n + 2$: Then $M = n + 2$, so (7) gives $(n+1)(n+3) = 3^{n!}$, also impossible.
- $y = n + 3$: Then $M = (n+2)(n+3)$, so (7) gives

  $$(n+1)((n+2)(n+3)+1) = 3^{n!}.$$

  If $n = 3k - 1$, this equals $9k(3k^2 + 3k + 1)$. Since $3k^2 + 3k + 1 \equiv 1 \pmod 3$, the right-hand side cannot be a pure power of 3. Contradiction.

Thus Case 2 yields no solutions. The only solutions are

$$(x, y, n) \in \{(0, 2, 1), (2, 0, 1), (1, 2, 1), (2, 1, 1)\}.$$

□

Solution to the Problem G:
Define
$$S := AB + BA - A - B - I_n.$$
Since A, B are symmetric, S is symmetric.
Taking trace both sides
$$Tr(SS^*) = Tr(S^2) = Tr(A^2B - 2ABA + BA^2) = 0,$$
Thus $S = 0$ and $AB + BA = A + B + I_n$.
Now consider any $x \in \mathbb{R}^n$:
$$x^T(A^2 + B^2)x = \|Ax\|^2 + \|Bx\|^2 \geq 0.$$
Moreover, $x^T(A^2 + B^2)x = 0$ if and only if $Ax = 0$ and $Bx = 0$. Hence
$$\ker(A^2 + B^2) = \ker A \cap \ker B.$$
Suppose $x \in \ker A \cap \ker B$. Then $Ax = 0$ and $Bx = 0$, and using (1):
$$0 = (AB + BA)x = (A + B + I_n)x = 0 + x,$$
which gives $x = 0$. Therefore
$$\ker(A^2 + B^2) = \{0\}.$$
Thus $A^2 + B^2$ is invertible and
$$\text{rank}(A^2 + B^2) = n.$$
□

Solution to the Problem P:
We solve a much wider problem: Let a fair die has N faces with $c_0, c_1, \ldots, c_{N-1}$ points on its faces and let a random process consists of repeatedly tossing it until n consecutive c_0 are tossed. We will calculate the expected value of the total sum of points obtained.
Let E_n the expected value of the sum of points tossed until n consecutive c_0-es are obtained. We proceed by induction on n.
For the initial case let X_k the result of k-th throw and by N – the total number of tosses until first c_0 appears. We have $X_1, \ldots, X_{N-1} \in \{c_1, \ldots, c_{N-1}\}$ and $X_N = c_0$, so

$$\mathbb{E}(X_1 + \cdots + X_N \mid N = k) = (k-1)\frac{c_1 + \cdots + c_{N-1}}{N-1} + c_0\,.$$

The probability of $\mathbb{P}(N = k)$ can be calculated from Bernoulli's process i.e., $\mathbb{P}(N = k) = \frac{(N-1)^{k-1}}{N^k}$, hence

$$\begin{aligned}
\mathrm{E}_1 &= \sum_{k=1}^{\infty} \mathbb{E}(X_1 + \cdots + X_N \mid N = k)\mathbb{P}(N = k) \\
&= \sum_{k=1}^{\infty} \frac{c_1 + \cdots + c_{N-1}}{N-1} \cdot \frac{(k-1)(N-1)^{k-1}}{N^k} + c_0 \sum_{k=1}^{\infty} \frac{(N-1)^{k-1}}{N^k} \\
&= \frac{c_1 + \cdots + c_{N-1}}{N(N-1)} \cdot \frac{\frac{N-1}{N}}{\left(1 - \frac{N-1}{N}\right)^2} + \frac{c_0}{N} \cdot \frac{1}{\left(1 - \frac{N-1}{N}\right)} \\
&= c_0 + c_1 + \cdots + c_{N-1}\,,
\end{aligned}$$

Now let H_n denote a random variable equal to the sum of points tossed in the process where n consecutive "c_0"-es are obtained. To obtain $n+1$ consecutive "c_0"-es we need first to have n consecutive "c_0"-es. The sum of tossed points during this part equals to the random variable H_n. Then either with probability $\frac{1}{N}$ the next "c_0" appears (so the process stops and we have the final sum $H_n + c_0$), or with the probability $\frac{N-1}{N}$ we restart the process from the beginning, remembering the number of points H_n plus those just tossed, that we have already counted. Hence we get the equality

$$\begin{aligned}
\mathrm{E}_{n+1} &= \mathbb{E}(H_{n+1}) = \frac{1}{N}\left(\mathbb{E}(H_n) + c_0\right) + \frac{N-1}{N}\left(\mathbb{E}(H_n) + \frac{c_1 + \cdots + c_{N-1}}{N-1} + \mathbb{E}(\tilde{H}_{n+1})\right) \\
&= \mathrm{E}_n + \frac{c_0 + \cdots + c_{N-1}}{N} + \frac{N-1}{N}\mathrm{E}_{n+1}\,,
\end{aligned}$$

where $\tilde{H}_{n+1}$ is a new random variable with the same distribution (and the same expected value) as H_{n+1} due to the independence of tosses. So finally $\mathrm{E}_{n+1} = N\,\mathrm{E}_n + (c_0 + \cdots + c_{N-1})$. Solving this simple recurrence gives

$$\mathrm{E}_n = (N^{n-1} + \cdots + N + 1)(c_0 + \cdots + c_{N-1}) = \frac{N^n - 1}{N-1}(c_0 + \cdots + c_{N-1})\,.$$

In our case, with $N = 6$ and $c_0 + \cdots + c_{N-1} = 1 + \cdots + 6 = 21$, we finaly get

$$\mathrm{E}_n = \frac{21}{5}(6^n - 1)\,.$$

□