PROBLEMS for team part

Problem A1 [*Proposed by* Pirmyrat Gurbanov & Murat Chashemov *from* International University for the Humanities and Development (Turkmenistan)]
Let $(R, +, \cdot)$ be a commutative ring with identity and let I, J be ideals of R. If R is a Principal Ideal Domain and $R = I + J$, prove that $I \cdot J = I \cap J$.
Remark
An ideal I of a commutative ring R is a subset $I \subset R$ such that it is closed under addition, and $a \cdot b \in I$ for all $a \in I$ and $b \in R$.
The product of two ideals I and J is by definition equal to

$$I \cdot J = \left\{ \sum_{k=1}^{n} a_k \cdot b_k \;\middle|\; a_k \in I,\ b_k \in J,\ n \in \mathbb{N} \right\}.$$

Problem A2 [*Proposed by* Robert Skiba *from* Nicolaus Copernicus University in Toruń (Poland)]
Let G be a finite group that is not commutative. Show that

$$P(G) = \frac{|\{(x, y) \in G \times G : xy \neq yx\}|}{|G \times G|} \geq \frac{3}{8}.$$

Problem C1 [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Does there exist a bijection $f \colon \mathbb{N} \to \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\, f(n)} \in \mathbb{Q}$?

Problem C2 [*Proposed by* Robert Skiba *from* Nicolaus Copernicus University in Toruń (Poland)]
Let $\Omega = \{z \in \mathbb{C} \colon -1 < \operatorname{Re} z < 1\}$ and $\mathcal{S}$ be the set of all analytic functions $f \colon \Omega \to \mathbb{C}$ satisfying $|f(z)| < 1$ for all $z \in \Omega$ and $f(0) = 0$. Evaluate $\sup\limits_{f \in \mathcal{S}} |f(i)|$.

Problem E1 [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Find all positive x such that $\left(\dfrac{1}{2}\right)^{x} + x^{1/2} = \sqrt{2}$.

Problem E2 [*Proposed by* Thomas Zürcher *from* University of Silesia in Katowice (Poland)]
Are there noncontinuous functions $f \colon (0, \infty) \to (0, \infty)$ such that

$$f(x + y) = \frac{f(x) f(y)}{f(x) + f(y)},$$

for all $x, y \in (0, \infty)$?

Problem G1 [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Triangle $\triangle ABC$ can be fully covered by 2025 discs of diameter 2. Determine whether it can be covered by 8100 discs of diameter 1.

Problem G.2 [*Proposed by* Leszek Pieniążek *from* Jagiellonian University, Kraków (Poland)]
Let $n \geq 2$ and $k < n$ be positive integers, and let $U, V_i \subset \mathbb{R}^n$ be linear subspaces of dimensions $\dim U = k$ and $\dim V_i = n - 1$ for $i = 1, 2, \ldots, n$ respectively. Let M be the $n \times n$ matrix with entries $m_{ij} = \dim(V_i + U) \cap V_j + \dim V_i \cap U$. Find all possible values of determinant $\det M$.

Problem P.1 [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Let $\xi_0 = 1$, and let $\sigma_n = \displaystyle\sum_{k=0}^{n} \xi_k$, where ξ_n are chosen at random uniformly from $[0, \sigma_{n-1}]$ for all $n = 1, 2, \ldots$.
Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\xi_n}{2^n}$ converges with probability 1 and calculate its expected value.

Problem P2 [*Proposed by* Marcin J. Zygmunt *from* University of Silesia, Katowice (Poland)]
Dirk is exploring an abandoned mine. The tunnels form an infinite regular tree, in which every node, including the root node that serves as the mine's entrance, is of degree d, where $d \geqslant 2$. He begins a random walk at the root node. Then, at each node, he chooses one of the d available tunnels with equal probability, including the tunnel he used to arrive at that node. He continues this process until he returns to his starting point at the root node, i.e. to the exit from the mine.

What is the probability that Dirk will eventually return to his starting point in a finite number of steps?

Excerpt from the Rules

$\vdots$

&7 Problems

$\vdots$

7. Solutions are evaluated (both in the individual and team part) in the following scale: $0, 3, 8, 10$ points, where:

a) 10 points are awarded for a complete solution (even with minor errors);

b) 8 points are awarded for a solution that is basically correct but contains major faults (e.g. a calculation error that simplified the reasoning, lack of substantial justification, etc.);

c) 3 points are awarded for a solution that is incomplete but contains a major step towards a correct solution;

d) 0 points are awarded to every other solution, even partial.

8. Solutions should be formulated in a clear, precise and legible manner that excludes ambiguity; unclear, imprecise or difficult to read solutions may result in a reduction in points, down to 0 points.

9. Solutions that merely cite statements, lemmas or theorems from the scientific literature will not be accepted. The Competition is not a contest of encyclopaedic knowledge, its aim is to test one's own ability to solve mathematical problems and to present a complete reasoning.

$\vdots$

SOLUTIONS

Solution to the Problem A1:
Let $x \in I \cdot J$. By definition of the product of ideals, we can write

$$x = \sum_{k=1}^{n} a_k \cdot b_k,$$

where $a_k \in I$ and $b_k \in J$.
Since I is an ideal, $a_k \in I$ implies $a_k \cdot b_k \in I$ (because ideals are closed under multiplication with elements of R). Similarly, J is an ideal, so $b_k \in J$ implies $a_k \cdot b_k \in J$.
Thus, $a_k \cdot b_k \in I \cap J$ for all k. Since $I \cap J$ is closed under addition, it follows that

$$x = \sum_{k=1}^{n} a_k \cdot b_k \in I \cap J.$$

Therefore, $I \cdot J \subseteq I \cap J$.

Let $t \in I \cap J$. Then $t \in I$ and $t \in J$.
Since R is a PID, the ideals I and J are principal. Let $I = (a)$ and $J = (b)$ for some $a, b \in R$. This means every element of I is of the form $r \cdot a$ for some $r \in R$, and every element of J is of the form $s \cdot b$ for some $s \in R$.
The condition $R = I + J$ implies that there exist $u \in I$ and $v \in J$ such that

$$1 = u + v.$$

Since $u \in I$ and $v \in J$, we can write $u = r \cdot a$ and $v = s \cdot b$ for some $r, s \in R$. Hence,

$$1 = r \cdot a + s \cdot b.$$

Now, consider $t \in I \cap J$. Since $t \in I$, we can write $t = t \cdot 1$. Substituting $1 = r \cdot a + s \cdot b$ gives

$$t = t \cdot (r \cdot a + s \cdot b) = (t \cdot r) \cdot a + (t \cdot s) \cdot b.$$

The term $(t \cdot r) \cdot a \in I \cdot J$ because $t \cdot r \in J$ and $a \in I$. Similarly, $(t \cdot s) \cdot b \in I \cdot J$ because $t \cdot s \in I$ and $b \in J$.
Thus, $t \in I \cdot J$, and it follows that $I \cap J \subseteq I \cdot J$.
Since $IJ \subseteq I \cap J$ and $I \cap J \subseteq IJ$, we conclude that

$$I \cdot J = I \cap J.$$

□

Solution to the Problem A2:
Note first that

$$|\{(x,y)\in G\times G : xy\neq ba\}| = |G|^2 - |\{(x,y)\in G\times G : xy=yx\}|.$$

So it suffices to bound from above

$$\frac{|\{(x,y)\in G\times G : xy=yx\}|}{|G|^2}.$$

Let

$$C_G(x)=\{y\in G : xy=yx\},\quad Z(G)=\{x\in G : xy=yx \text{ for all } y\in G\}.$$

For a non-abelian group G, we use the following standard facts (with proofs for completeness):

- **If $x\notin Z(G)$, then $|G:C_G(x)|\geq 2$, hence $|C_G(x)|\leq |G|/2$.** Consider the conjugation action of G on itself: $g\cdot x = gyg^{-1}$. The stabilizer of x is exactly $C_G(x)$, while the orbit is the conjugacy class $\mathrm{Cl}(x)=\{gxg^{-1} : g\in G\}$. By the orbit–stabilizer formula,

  $$|\mathrm{Cl}(x)| = |G:C_G(x)|.$$

  If $a\notin Z(G)$, there exists $g\in G$ with $gx\neq xg$, so $gxg^{-1}\neq x$ and $|\mathrm{Cl}(x)|\geq 2$. Hence $|G:C_G(x)|\geq 2$, i.e. $|C_G(x)|\leq |G|/2$.

- **Since G is non-abelian, $G/Z(G)$ is not cyclic; in particular $|G:Z(G)|\geq 4$.** Suppose for contradiction that $G/Z(G)$ is cyclic, say $G/Z(G)=\langle gZ(G)\rangle$. Then every $x\in G$ can be written $x=zg^k$ with some $z\in Z(G)$ and $k\in\mathbb{Z}$. For arbitrary $x=z_1g^i$ and $y=z_2g^j$ we get

  $$xy = z_1z_2g^{i+j} = z_2z_1g^{j+i} = yx,$$

  because z_1, z_2 are central and powers of g commute. Thus G would be abelian, a contradiction. Therefore $G/Z(G)$ is not cyclic. Since every group of order $1,2,3$ is cyclic, we must have $|G:Z(G)|\notin\{1,2,3\}$, hence $|G:Z(G)|\geq 4$, i.e. $|Z(G)|/|G|\leq 1/4$.

Moreover,

$$\begin{aligned}
\frac{|\{(x,y)\in G\times G : xy=yx\}|}{|G|^2} &= \frac{1}{|G|^2}\sum_{a\in G}|C_G(a)| \\
&= \frac{1}{|G|^2}\Big(|Z(G)|\cdot|G| + \sum_{x\in G\setminus Z(G)}|C_G(x)|\Big) \\
&= \frac{1}{|G|}\left(|Z(G)| + \sum_{x\notin Z(G)}\frac{|C_G(x)|}{|G|}\right) \\
&\leq \frac{1}{|G|}\left(|Z(G)| + \frac{|G|-|Z(G)|}{2}\right) \\
&= \frac{1}{2}\left(1+\frac{|Z(G)|}{|G|}\right) \\
&\leq \frac{1}{2}\left(1+\frac{1}{4}\right) = \frac{5}{8}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(G) &= 1 - \frac{|\{(x,y)\in G\times G : xy=yx\}|}{|G|^2} \\
&\geq 1-\frac{5}{8} = \frac{3}{8}.
\end{aligned}$$

$\square$

Solution to the Problem C1:
The answer is affermative. See for instance the function

$$f(n)=\begin{cases}2, & \text{for } n=1;\\ n+2, & \text{for even } n;\\ n-2, & \text{for odd } n>1.\end{cases}$$

We will show that f satisfies problem's condition. Indeed

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n\,f(n)} =& \frac{1}{1\cdot 2}+\sum_{k=1}^{\infty}\frac{1}{2k\,(2k+2)}+\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k-1)}\\ =&\frac{1}{2}+\frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)\\ =&\frac{1}{2}+\frac{1}{4}+\frac{1}{2}=1\in\mathbb{Q}\,.\end{aligned}$$

□

Solution to the Problem C2:
A standard tool is the following

Schwarz Lemma
If $F : \mathbf{D} \to \mathbf{D}$ is holomorphic with $F(0) = 0$, then $|F(w)| \le |w|$ for all $w \in \mathbf{D}$.
We map Ω onto the unit disk. Define

$$g(z) = e^{\frac{\pi i}{2} z}, \qquad h(w) = \frac{w-1}{w+1}, \qquad \varphi = h \circ g.$$

For $z = x + iy$ with $-1 < x < 1$, we have $g(z) = e^{\dfrac{\pi i}{2} x - \dfrac{\pi}{2} y}$, which lies in the right half-plane $\{w : \mathrm{Re}\, w > 0\}$. The map $h(w) = \frac{w-1}{w+1}$ sends the right half-plane onto the unit disk $\mathbf{D}$. Indeed, if $w = x + iy$ then

$$|h(w)| < 1 \iff |w-1| < |w+1| \iff (x-1)^2 + y^2 < (x+1)^2 + y^2 \iff x > 0,$$

so Re $w > 0$ corresponds exactly to $|h(w)| < 1$. Moreover $h^{-1}(z) = \dfrac{1+z}{1-z}$ shows bijectivity.
Thus $\varphi : \Omega \to \mathbf{D}$ is a map with $\varphi(0) = 0$. For any $f \in \mathcal{S}$, define $F = f \circ \varphi^{-1} : \mathbf{D} \to \mathbf{D}$. Then $F(0) = 0$, so by Schwarz' lemma,

$$|F(w)| \le |w|, \quad w \in \mathbf{D}.$$

In particular,

$$|f(z)| = |F(\varphi(z))| \le |\varphi(z)|, \quad z \in \Omega.$$

Therefore

$$\sup_{f \in \mathcal{S}} |f(i)| = |\varphi(i)|.$$

Finally, compute:

$$g(i) = e^{\dfrac{\pi i}{2} i} = e^{-\dfrac{\pi}{2}} =: t, \quad \varphi(i) = h(t) = \frac{t-1}{t+1} = -\frac{1-t}{1+t}.$$

So

$$|\varphi(i)| = \frac{1 - e^{-\pi/2}}{1 + e^{-\pi/2}} = \tanh\left(\frac{\pi}{4}\right).$$

Answer:

$$\sup_{f \in \mathcal{S}} |f(i)| = \tanh\left(\frac{\pi}{4}\right).$$

The supremum is attained by the extremal functions $f(z) = e^{i\theta} \varphi(z)$, $\theta \in \mathbb{R}$.

□

Solution to the Problem E1:
Direct calculation shows that $x = \dfrac{1}{2}$ is a solution. Indeed,

$$\left(\frac{1}{2}\right)^{1/2} + \left(\frac{1}{2}\right)^{1/2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}\,.$$

To prove that this solution is unique, we shall show that the function $f(x) = \left(\dfrac{1}{2}\right)^x + x^{1/2}$ is strictly increasing for all $x > 0$. The derivative of f is equal to

$$f'(x) = \left(\frac{1}{2}\right)^x \ln\frac{1}{2} + \frac{1}{2x^{1/2}} = -\left(\frac{1}{2}\right)^x \ln 2 + \frac{1}{2\sqrt{x}}\,,$$

so the function f is strictly increasing if $f'(x) > 0$ for all $x > 0$. This condition is equivalent to the inequality

$$\frac{1}{2\sqrt{x}} > \left(\frac{1}{2}\right)^x \ln 2,$$

which can be rearranged as

$$1 > \sqrt{x}\, 2^{1-x} \ln 2. \tag{26.1}$$

Since the right-hand side of (26.1) is positive for $x > 0$, we may prove the inequality by showing that the square of the right-hand side is always less than 1.
Let

$$g(x) = \left(\sqrt{x}\, 2^{1-x} \ln 2\right)^2 = x\, 4^{1-x} (\ln 2)^2.$$

We note that both $\lim\limits_{x\to 0^+} g(x) = 0$ and $\lim\limits_{x\to+\infty} g(x) = 0$. As $g(x)$ is positive for $x \in (0, \infty)$, it must attain a global maximum at a critical point. Differentiating gives

$$g'(x) = 4^{1-x} (\ln 2)^2 (1 - x \ln 4).$$

Setting $g'(x) = 0$ yields the unique critical point $a_0 = \frac{1}{\ln 4} = \log_4 e < 1$. The maximum value of g is therefore

$$g(a_0) = a_0\, 4^{1-a_0} (\ln 2)^2 = \frac{4^{1-\log_4 e} (\ln 2)^2}{2 \ln 2} = \frac{4}{2e} \ln 2 < 1$$

as both factors are less than 1. This implies that the right-hand side of inequality (26.1) is also less than 1 for all $x > 0$. Therefore, the function f is strictly increasing. A strictly increasing function can assume any given value at most once, so the solution $x = \dfrac{1}{2}$ is unique. $\square$

Solution to the Problem E2:
Plugging in $x = y$, we obtain

$$f(2x) = f(x+x) = \frac{f(x)^2}{2f(x)} = \frac{1}{2}f(x).$$

With $x = 2x$ and $y = x$, we have

$$f(3x) = f(2x+x) = \frac{f(2x)f(x)}{f(2x)+f(x)} = \frac{\frac{1}{2}f(x)^2}{\frac{1}{2}f(x)+f(x)} = \frac{\frac{1}{2}f(x)^2}{\frac{3}{2}f(x)} = \frac{1}{3}f(x).$$

We claim that for $n \in \mathbb{N}$, we have

$$f(nx) = \frac{1}{n}f(x).$$

We prove the statement via induction over n. Having it clearly for $n = 1$, we assume that it holds for some $n \in \mathbb{N}$. Then, plugging in $x = nx$ and $y = x$:

$$f((n+1)x) = f(nx+x) = \frac{f(nx)f(x)}{f(nx)+f(x)} = \frac{\frac{1}{n}f(x)^2}{\left(\frac{1}{n}+1\right)f(x)} = \frac{\frac{1}{n}f(x)}{\frac{n+1}{n}} = \frac{1}{n+1}f(x).$$

Thus, we have $f(nx) = \frac{1}{n}f(x)$ for all $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}$. Replacing x by x/n:

$$f(x) = \frac{1}{n}f\left(\frac{x}{n}\right),$$

and thus for all $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}$

$$f\left(\frac{x}{n}\right) = nf(x).$$

Furthermore,

$$f\left(\frac{m}{n}x\right) = \frac{1}{m}f\left(\frac{1}{n}x\right) = \frac{n}{m}f(x).$$

For each $q \in \mathbb{Q}_+^*$, we have

$$f(q) = \frac{1}{q}f(1).$$

Let us show now that f is monotone decreasing. Assume that $x, y \in \mathbb{R}_+^*$ with $x < y$. Then

$$f(y) = f(x+y-x) = \frac{f(x)f(y-x)}{f(x)+f(y-x)},$$

so

$$\frac{f(y)}{f(x)} = \frac{f(y-x)}{f(x)+f(y-x)} \leq 1,$$

giving the monotonicity.
Fix $x_0 \in \mathbb{R}_+^*$. We approximate it by two sequences $(q_n)_n$ and $(r_n)_n$ of rational numbers, the first one being strictly increasing approaching x_0 from the left, and the second one being strictly decreasing approaching x_0 from the right. Then

$$\frac{1}{r_n}f(1) = f(r_n) \leq f(x_0) \leq f(q_n) = \frac{1}{q_n}f(1).$$

Taking the limit as n approaches 0 gives

$$f(x_0) = \frac{1}{x_0}f(1).$$

So, all solutions are continuous.
We even can give all the solutions. Note that if $c \in \mathbb{R}_+^*$, then $f(x) = \frac{c}{x}$ is a solution:

$$f(x+y) = \frac{c}{x+y}$$

and

$$\frac{f(x)f(y)}{f(x)+f(y)} = \frac{\frac{c^2}{xy}}{\frac{c}{x}+\frac{c}{y}} = \frac{\frac{c^2}{xy}}{\frac{cy+cx}{xy}} = \frac{c}{x+y}.$$

We are done. $\square$

Solution 2.:
We assume first that f is a solution of the equation (leaving it open for now whether the solution is continuous). As the target does not include 0, the function $g := 1/f$ is well-defined. Furthermore, it is clear that the target is $\mathbb{R}_+^*$ as well. We have

$$g(x+y) = \frac{1}{f(x+y)} = \frac{f(x)+f(y)}{f(x)f(y)},$$

and to ease our mind, we note that the denominator does not vanish. Hence,

$$g(x+y) = \frac{1}{f(y)} + \frac{1}{f(x)} = g(y) + g(x).$$

We want to show that g has to be continuous. First, we show that $g(p/q) = pg(1)/q$ for $p, q \in \mathbb{N}$. We start with arguing that $g(n) = ng(1)$ for all $n \in \mathbb{N}$. We do so by induction over n. The claim being trivial for $n = 1$, we assume that it holds for some $n \in \mathbb{N}$. Then,

$$g(n+1) = g(n) + g(1) = ng(1) + g(1) = (n+1)g(1).$$

By induction, we see that for all $n \in \mathbb{N}$ and $a_k \in \mathbb{R}_+^*$ for $k \in \{1, \dots, n\}$,

$$\sum_{k=1}^{n} g(a_k) = g\left(\sum_{k=1}^{n} a_k\right).$$

Now, we show that $g(1/n) = g(1)/n$ for $n \in \mathbb{N}$, this is $ng(1/n) = g(1)$ for all $n \in \mathbb{N}$. But, by the above

$$ng(1/n) = \sum_{k=1}^{n} g(1/n) = g\left(\sum_{k=1}^{n} \frac{1}{n}\right) = g(1).$$

Finally,

$$g\left(\frac{p}{q}\right) = g\left(\sum_{k=1}^{p} \frac{1}{q}\right) = \sum_{k=1}^{p} g\left(\frac{1}{q}\right) = pg\left(\frac{1}{q}\right) = \frac{p}{q} g(1).$$

Next, we argue that g is monotone. For this, it is crucial that g does not attain negative values. Let $x, y \in \mathbb{R}_+^*$ with $x < y$. Then,

$$g(y) = g(x + (y-x)) = g(x) + g(y-x) \geq g(x) + 0 = g(x).$$

Assume now that g is not continuous at some point $x_0 \in \mathbb{R}^*$. We approximate it by two sequences $(q_n)_n$ and $(r_n)_n$ of rational numbers, the first one being strictly increasing approaching x_0 from the left, and the second one being strictly decreasing approaching x_0 from the right. Thus,

$$q_n g(1) \leq g(x_0) \leq r_n g(1),$$

and taking the limit, we see that $g(x_0) = x_0 g(1)$. In conclusion, there is some constant $c \geq 0$ such that $g(x) = cx$ for all $x \in \mathbb{R}$. Hence, $f(x) = \frac{c'}{x}$ for some constant $c' \geq 0$, and f is thus continuous. □

Solution to the Problem G1:
Yes, the answer is affermative. Let K, L, M be the midpoints of the sides of triangle $\triangle ABC$ (see the figure)

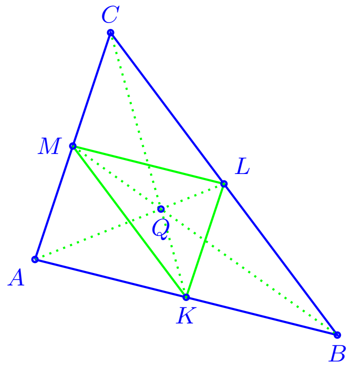

Figure 1: Figure to the Problem 33

The triangles $\triangle AKM$, $\triangle BKL$, and $\triangle CLM$ are the images of $\triangle ABC$ under homotheties with a scale factor of $\frac{1}{2}$, centered at the corresponding vertices A, B, and C. Furthermore, triangle $\triangle KLM$ is the image of $\triangle ABC$ under a homothety with a scale factor of $-\frac{1}{2}$, centered at the centroid Q of $\triangle ABC$.
Therefore, each of the small triangles can be covered by 2025 discs of radius 1, which are images of the original discs covering $\triangle ABC$. Hence, the entire triangle $\triangle ABC$ can be covered by 8100 discs of radius 1.

$\square$

Solution to the Problem G2:

Lemma (39).1
For any linear subspaces $V_1, V_2, V_3 \subset \mathbb{R}^n$ *and* (k, l, m)*: a permutation of* $(1, 2, 3)$ *define*

$$d_k = \dim V_k \cap (V_l + V_m) + \dim V_l \cap V_m.$$

Equality $d_1 = d_2 = d_3$ *holds.*
Proof: One can check, that the common value is

$$d_k = \dim V_k \cap V_l + \dim V_k \cap V_m + \dim V_l \cap V_m - \dim V_k \cap V_l \cap V_m.$$

◇

Using the lemma reformulates definition of matrix M in more symmetrical form

$$m_{ij} = \dim(V_i + V_j) \cap U + \dim V_i \cap V_j.$$

Case $V_i = V_j$**.** Rows number i and j are equal, so $\det M = 0$.
Case $V_i \neq V_j$ **for any** i, j**.** Obviously $V_i + V_j = \mathbb{R}^n$ and $\det V_i \cap V_j = n - 2$ for $i \neq j$. So every number out of the diagonal is equal $k + n - 2$.
Further $m_{ii} = \begin{cases} k+n-1 & \text{if } U \subset V_i \\ k+n-2 & \text{if } U \not\subset V_i \end{cases}$. For any j it is possible to choose V_i in such a way that precisely j of them fulfil $U \subset V_i$, so every number on diagonal can be of those two forms independently. But if there are at least two numbers $k + n - 2$, then there are two equal rows, so determinant is 0.
Below, using elementary operations on rows shows how to get result. Let $a = k + n - 2$.

$$\det \begin{bmatrix} a{+}1 & a & \dots & a & a \\ a & a{+}1 & \dots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & \dots & a{+}1 & a \\ a & a & \dots & a & a{+}1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ a & a & \dots & a & a{+}1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & na{+}1 \end{bmatrix} = na{+}1;$$

$$\det \begin{bmatrix} a+1 & a & \dots & a & a \\ a & a+1 & \dots & a & a \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & \dots & a+1 & a \\ a & a & \dots & a & a \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ a & a & \dots & a & a \end{bmatrix} = a.$$

The possible results are: 0, $k + n - 2$, $n(k + n - 2) + 1$. □

Solution to the Problem P1:
We have

$$\mathbb{E}(\xi_n|\sigma_{n-1}) \;=\; \mathbb{E}(\xi_n|\xi_{n-1},\xi_{n-2},\dots,\xi_1) = \frac{\sigma_{n-1}}{2}$$

for $n \geqslant 1$. On the other hand $\sigma_n = \sigma_{n-1} + \xi_n$, hence

$$\mathbb{E}(\sigma_n|\sigma_{n-1}) = \sigma_{n-1} + \mathbb{E}(\xi_n|\sigma_{n-1}) = \frac{3}{2}\sigma_{n-1}\,.$$

So

$$\mathbb{E}(\xi_n) = \mathbb{E}\Big(\mathbb{E}(\xi_n|\sigma_{n-1})\Big) = \frac{\mathbb{E}(\sigma_{n-1})}{2} = \frac{1}{2}\,\frac{3}{2}\mathbb{E}(\sigma_{n-2}) = \frac{3^{n-1}}{2^n}\mathbb{E}(\sigma_0) = \frac{3^{n-1}}{2^n}\,.$$

Thus (all variables are non negative, so we can put out summation by Tonelli's Theorem)

$$\mathbb{E}\left(\sum_{n=0}^{\infty}\frac{\xi_n}{2^n}\right) \;=\; \xi_0 + \sum_{n=1}^{\infty}\frac{\mathbb{E}(n)}{2^n} \;=\; 1 + \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \;=\; 2\,,$$

which shows also that the series converges with probability 1. $\square$

Solution to the Problem P2:
The problem can be modeled as a one-dimensional random walk on the set of non-negative integers $\mathbb{Z}_+\{0,1,2,\dots\}$, where the position corresponds to Dirk's distance (depth) from the root. At any depth $m>0$, the probability of moving one step closer to the root (so to the left) is $p=\frac{1}{d}$, and the probability of moving one step further away (to the right) is $1-p=\frac{d-1}{n}$.
Let A denote the event of eventually returning to the starting point, and let A_n be the event that the first return to the starting point occurs after n steps. Clearly, n must be even. To derive the probability of A_n, we note that a path returning to the root for the first time at step $n=2k$ must have spent the previous $n-2$ steps returning to the first node it visited, without hitting the root, before taking the final step back to the origin. The number of such paths is related to the Catalan numbers. Hence the probability of a first return at step n is therefore given by

$$\mathbb{P}(A_n)=\begin{cases}0, & n=0 \text{ or } n \text{ is odd},\\ C_{k-1}p^k(1-p)^{k-1}, & n=2k, k\geqslant 1.\end{cases}$$

The following lemma will be required.
Lemma (47).1
Let $f(x)=\sum_{n=0}^{\infty}C_nx^n$, *where* C_n *is the* n*-th Catalan's number. Then*

$$f(x)=\frac{1-\sqrt{1-4x}}{2x}$$

and the series converges for $|x|<\frac{1}{4}$.
Proof: The proof uses the recurrence relation for Catalan numbers,

$$C_{n+1}=\sum_{k=0}^{n}C_kC_{n-k}\,.$$

It follows that

$$f(x)^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}C_kC_{n-k}\right)x^n=\sum_{n=0}^{\infty}C_{n+1}x^n=\frac{f(x)-C_0}{x}\,.$$

Since $C_0=1$, this yields the quadratic equation $x\,f(x)^2-f(x)+1=0$ as $C_0=1$. Solving it for f gives two potential solutions $f(x)=\dfrac{1\pm\sqrt{1-4x}}{2x}=\dfrac{2}{1\mp\sqrt{1-4x}}$. However, the function f converges at $x=0$, which is only satisfied by choosing the negative sign. Thus

$$f(x)=\frac{1-\sqrt{1-4x}}{2x}\,.$$

◇

We can now calculate the total probability

$$\begin{aligned}\mathbb{P}(A)&=\sum_n\mathbb{P}(A_n)=\sum_{k=1}^{\infty}C_{k-1}p^k(1-p)^{k-1}\\ &=p\sum_{n=0}^{\infty}C_n\,(p(1-p))^n=p\,f(p(1-p))=\frac{1-\sqrt{1-4p(1-p)}}{2(1-p)}\,.\end{aligned}$$

Noting that $1-4p(1-p)=1-4p+4p^2=(1-2p)^2$ yields

$$\mathbb{P}(A)=\frac{1-|1-2p|}{2(1-p)}=\begin{cases}\frac{p}{1-p}, & \text{for } 1>2p,\\ 1, & \text{for } 1\leqslant 2p.\end{cases}$$

In our case $p=\dfrac{1}{d}$, where $n\geqslant 2$, therefore the probability that Dirk eventualy returns to the starting point is

$$\mathbb{P}(A)=\frac{\frac{1}{d}}{\frac{d-1}{d}}=\frac{1}{d-1}\,.$$

Remark
The other solutions based on recurrence has a fundamental flaw: they lead to a quadratic equation of the form $nP(A)^2-(d-1)P(A)+1=0$, *which has two solutions,* $P(A)=1$ *or*

$P(A) = \frac{1}{d-1}$. *There is a precise mathematical argument to decide which of these two solutions is correct, but I do not report it here.*

$\square$